Topics: - Initial Value Problems
(using the Laplace Transform)

General Strategy

Why solve an IVP using the Laplace Transform.
Here are three main steps:

1. Take the L.T of the problem.
2. Use properties of the L.T and the initial data, provided by hand, to solve for the L.T of the solution.
3. Transform back (the hard part) the found expression to obtain the "true" solution as a function of the original variable t.

That's to be easy example to see this.

Ex: Solve the IVP

$$y'' - 2y' + 5y = -8e^t,$$
$$y(0) = 2$$
$$y'(0) = 12$$

Step 1: We first use linearity of the LT and the following identities:

$$\mathcal{L}\{y''(t)\}(s) = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0),$$
$$\mathcal{L}\{y(t)\}(s) = Y(s), \quad \text{and}$$
$$\mathcal{L}\{-8e^t\} = -8 \cdot \frac{1}{s-1}.$$

If we apply this to our IVP, we have:

$$\mathcal{L}\{y'' - 2y' + 5y\} = s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 5(Y(s))$$
$$= s^2Y(s) - 2s - 12 - 2sY(s) + 4 + 5Y(s)$$

and this must be equal to:

$$\frac{-8}{s-1}.$$

Step 2: Now we do some algebra to solve for $Y(s)$:

$$s^2Y(s) - 2s - 12 - 2sY(s) + 4 + 5Y(s) = \frac{-8}{s-1}$$
$$\Rightarrow Y(s)[s^2 - 2s + 5] = \frac{-8}{s-1} + 2s + 8$$
$$\Rightarrow Y(s) = \frac{1}{s^2-2s+5}\left(2s + 8 - \frac{8}{s-1}\right)$$
$$= \frac{1}{s^2-2s+5}\left(2(s+4)(s-1) - 8\right)$$
$$= \frac{2s^2+10s}{(s^2-2s+5)(s-1)}.$$

Step 3: Now we must target $Y(s)$ to obtain $y(t)$. Using partial fractions gives us:

$$\frac{2s^2+10s}{(s^2-2s+5)(s-1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s-1}$$
$$\Rightarrow \underbrace{(As+B)(s-1)}_{\to 2} + \underbrace{C(s^2-2s+5)}_{\to 0} = 2s^2+10s$$
$$\Rightarrow \underbrace{(A+C)s^2}_{\to 2} + \underbrace{(A+B-2C)s}_{\to 0} + \underbrace{(B+5C)}_{\to 0} = 2s^2+10s$$

$$\Rightarrow B = -5C \quad \Rightarrow B = -5(2-A)$$
$$\Rightarrow C = 2-A \quad = 5A - 10.$$
$$\Rightarrow 10 = A + B - 2C$$
$$= A + (5A-10) - 2(2-A)$$
$$= A + 5A + 2A - 14$$
$$\Rightarrow \boxed{A = \frac{24}{8} = 3}, \text{ so } \boxed{C = 2 - 3 = -1}$$
$$\text{and } B = -5(-1) = 5.$$

Thus,

$$\frac{2s^2+10s}{(s^2-2s+5)(s-1)} = \frac{3s+5}{(s^2-2s+5)} - \frac{1}{s-1}.$$

Now we massage this into a more familiar form. First, complete the square in the denominator:

$$s^2 - 2s + 5 = (s-1)^2 + 2^2.$$

Then,

$$Y(s) = \frac{3s+5}{(s-1)^2+2^2} - \frac{1}{s-1}$$
$$= \frac{3(s-1)+3+5}{(s-1)^2+2^2} - \frac{1}{s-1}$$
$$= 3\left[\frac{s-1}{(s-1)^2+2^2}\right] \quad \text{Looks like shifted cosine}$$
$$+ 4\left[\frac{2}{(s-1)^2+2^2}\right] \quad \text{Looks like shifted sine}$$
$$- \frac{1}{s-(1)} \quad \text{Exponential}$$

So, we finally have:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t)$$
$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-(1)}\right\}$$
$$y(t) = 3e^t\cos(2t) + 4e^t\sin(2t) - e^t,$$

and this is our final solution.

Now that we understand the general procedure, let's try a more difficult problem.

Ex: Solve

$$y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi),$$
$$y(0) = 1,$$
$$y'(0) = 1.$$

Recall: $u(t-a)$ is the Heaviside function:

$$u(t-a) := \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Step 1: Take the LT of both sides:

$$\mathcal{L}\{y'' + 2y' + 2y\} = s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s)$$
$$= Y(s)[s^2 + 2s + 2] - s - 3.$$

Then, also recall that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

So, $\mathcal{L}\{u(t-2\pi) - u(t-4\pi)\}$

$$= \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s}.$$

Step 2: Combine and solve for $Y(s)$:

$$Y(s)[s^2+2s+2] - (s+3) = \frac{e^{-2\pi s}(1 - e^{-2\pi s})}{s}$$
$$\Rightarrow Y(s) = \frac{e^{-2\pi s}(1-e^{-2\pi s})}{s(s^2+2s+2)} + \frac{s+3}{s^2+2s+2}$$

Step 3: Invert. We will have to use partial fractions again:

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$
$$\Rightarrow A(s^2+2s+2) + s(Bs+C) = 1$$
$$\Rightarrow \underbrace{(A+B)}_{\to 0}s^2 + \underbrace{(2A+C)}_{\to 0}s + \underbrace{(2A)}_{\to 2} = 1$$

So, $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -1$.

So,

$$\frac{e^{-2\pi s} - e^{-4\pi s}}{s(s^2+2s+2)} = \left(e^{-2\pi s} - e^{-4\pi s}\right)\left[\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1}\right] - \frac{1}{2}\frac{1}{(s+1)^2+1}\Bigg]$$
$$= \left(e^{-2\pi s} - e^{-4\pi s}\right)\left[\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right] = \left(e^{-2\pi s} - e^{-4\pi s}\right)\left[F_1(s) + F_2(s) + F_3(s)\right]$$

$$\left(-\frac{s}{2} - 1\right)\frac{1}{s^2+2s+2} = -\frac{1}{2}\frac{(s+1)}{(s+1)^2+1} + \left(\frac{1}{2} - 1\right)\frac{1}{(s+1)^2+1}$$
$$= -\frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

Also, recall that:

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a),}$$

where $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$.

So, we only need to determine

$$\frac{1}{2}\cdot\frac{1}{s}\left\{\frac{1}{s}\right\}, \ \frac{1}{2}\frac{s+1}{(s+1)^2+1}\Bigg\} \quad \text{and} \quad \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$F_1(s)$: $\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{1}{2}$,

$F_2(s)$: $\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = \frac{1}{2}e^{-t}\cos(t)$, $F_3(s)$: $\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \frac{1}{2}e^{-t}\sin(t)$

So, if $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$, then

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s}\right\} = u(t-2\pi),$$

and

$$\mathcal{L}^{-1}\left\{\frac{e^{-4\pi s}}{s}\right\} = u(t-4\pi).$$

To find we can write

$$\mathcal{L}^{-1}\{F_1(s) + F_2(s) + F_3(s)\} = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos(t) + \sin(t))$$

and so

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}(F_1 + F_2 + F_3)\right\} = u(t-2\pi)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}(\cos(t-2\pi) + \sin(t-2\pi))\right)$$

and similarly:

$$\mathcal{L}^{-1}\left\{e^{-4\pi s}(F_1 + F_2 + F_3)\right\} = u(t-4\pi)\left(\frac{1}{2} - \frac{1}{2}e^{(t-4\pi)}(\cos(t-4\pi) + \sin(t-2\pi))\right)$$

Finally, $\frac{s+3}{s^2+2s+2} = \frac{s+3}{(s+1)^2+1}$

$$= \frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1}$$

And so:

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)^2+1}\right\} = e^{-t}\cos(t) + 2e^{-t}\sin(t).$$

Our final solution is then:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \frac{u(t-2\pi)}{2}\left(1 - e^{-(t-2\pi)}(\sin(t) + \cos(t))\right)$$
$$+ \frac{u(t-4\pi)}{2}\left(1 - e^{(t-4\pi)}(\sin(t) + \cos(t))\right)$$
$$+ e^{-t}(\sin(t) + 2\cos(t))$$

Ex: Solve

$$y'' + 3y' + 2y = f(t),$$
$$y(0) = 0,$$
$$y'(0) = 0.$$

where $f(t)$ is the periodic function:

$f(t) = e^t$, $T = 1$.

i.e. $f(t)$ looks like:

First, recall that for a periodic function $g(t)$ with period T,

$$F(s) = \frac{F_T(s)}{1 - e^{-Ts}},$$

where $F_T(s) := \int_0^T e^{-st}g(t)\,dt$.

So, we'll compute $F_T(s)$ first:

$$F_T(s) = \int_0^1 e^{-st}e^t\,dt$$
$$= \int_0^1 e^{-t(s-1)}\,dt$$
$$= \frac{-1}{s-1}e^{-t(s-1)}\Big|_0^1$$
$$= -\frac{1}{s-1} + \frac{1}{s-1}$$
$$= \frac{1}{s-1}\left(1 - e^{-(s-1)}\right)$$

So, $F_T(s) = \frac{1 - e^{-(s-1)}}{(s-1)(1-e^{-s})}$, so?

Step 1: Take the LT of both sides:

$$\mathcal{L}\{y'' + 3y' + 2y\} = s^2Y(s) + 3sY(s) + 2Y(s) = \mathcal{L}\{f(t)\} = \frac{1 - e^{-(s-1)}}{(s-1)(1-e^{-s})}$$

Step 2: Solve for $Y(s)$:

$$Y(s) = \frac{1 - e^{-(s-1)}}{(s-1)(s^2+3s+2)(1-e^{-s})}$$

Step 3: Invert to find $y(t)$.
It will be useful to know that:

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{(s-1)(1-e^{-s})}$$

$$\Leftrightarrow g(t) = \sum_{n=0}^{\infty} e^{-n(t-1)}u_n(t)$$

Now, partial fractions:

$$\frac{1}{(s-1)(s^2+3s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+3s+2}$$
$$\Rightarrow A(s^2+3s+2) + (s-1)(Bs+C) = 1$$
$$\Rightarrow \underbrace{(A+B)}_{\to 0}s^2 + \underbrace{(3A+C-B)}_{\to 0}s + \underbrace{-C}_{\to 2} = 1$$

So, $C = -1$, $A = -B$, and so

$$0 = 3A + 1 + A$$
$$\Rightarrow A = -1/4, \text{ and}$$
$$B = 1/4.$$

So,

$$\frac{1}{(s-1)(s^2+3s+2)} = \frac{-1}{4(s-1)} + \frac{s-1}{s^2+3s+2}$$
$$= \frac{-1}{4(s-1)} + \frac{1}{4}\left(\frac{s-1}{(s+\frac{3}{2})^2-\frac{1}{4}}\right)$$
$$= \frac{-1}{4(s-1)} + \frac{1}{4}\left(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2-(\frac{1}{2})^2} - \frac{(1+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}\right)$$
$$= \frac{-1}{4(s-1)} + \frac{1}{4}\left(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}\right) - 5\left(\frac{1/2}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}\right)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{-1}{4}e^{t} \qquad \frac{1}{4}e^{-\frac{3t}{2}}\cosh(\tfrac{t}{2}) \qquad -5e^{-\frac{3t}{2}}\cosh(\tfrac{t}{2}).$$

To anyone who has made it this far... I'm impressed. Things get a little ridiculous from here on out. The main takeaway is how to transform a periodic function. To fully solve this problem, email me and we can try together.