Lab Nine 2

Wednesday, October 20, 2021   9:48 AM

Ex. $y' + 2xy = 0$ about $x_0 = 0$.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$. then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

Plug in:
$$0 = \sum_{n=1}^{\infty} a_n n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$
$$= \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

wrong powers of x!

$$= a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1) x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$
$$= a_1 + \sum_{n=0}^{\infty} a_{n+2}(n+2) x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$
$$0 = a_1 + \sum_{n=0}^{\infty} [a_{n+2}(n+2) + 2a_n] x^{n+1}$$

$\Rightarrow$ $a_1 = 0 = a_{n+2}(n+2) + 2a_n$

So, $a_{n+2} = \frac{-2}{n+2} a_n$

$\underline{n=0}$: $a_2 = \frac{-2}{2} a_0 = \frac{-1}{1} a_0$ $\underline{n=1}$: $a_3 = \frac{-2}{3} a_1 = 0$

$\underline{n=2}$: $a_4 = \frac{-2}{4} a_2 = \frac{-2}{4} \cdot \frac{-2}{2} \cdot a_0 = \frac{-1}{2} \cdot \frac{-1}{1} \cdot a_0$ $\underline{n=3}$: $a_5 = \frac{-2}{5} a_3 = 0$

$\underline{n=4}$: $a_6 = \frac{-2}{6} \cdot a_4 = \frac{-2}{6} \cdot \frac{-2}{4} \cdot \frac{-2}{2} \cdot a_0 = \frac{-1}{3} \cdot \frac{-1}{2} \cdot \frac{-1}{1} a_0$ $\underline{n=5}$: $a_7 = 0$

$\underline{n=6}$: $a_8 = \frac{-2}{8} a_6 = \frac{-2}{8} \cdot \frac{-2}{6} \cdot \frac{-2}{4} \cdot \frac{-2}{2} a_0 = \frac{-1}{4} \cdot \frac{-1}{3} \cdot \frac{-1}{2} \cdot \frac{-1}{1} \cdot a_0$

So, $a_{2n} = \frac{(-1)^n}{n!} a_0$, $a_{2n+1} = 0$

So, $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_2 x^2 + a_4 x^4 + \cdots$
$+ a_1 + a_3 x^3 + a_5 x^5 + \cdots$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_0 x^{2n} + 0$$
$$= a_0 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = a_0 e^{-x^2}$$

Ex. $(1-x)y' = y$

$$\Rightarrow (1-x) \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\Rightarrow a_1 \cdot 1 \cdot x^0 + \sum_{n=1}^{\infty} a_{n+1}(n+1) x^n - \sum_{n=1}^{\infty} a_n n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$
$$\Rightarrow (a_1 - a_0) + \sum_{n=1}^{\infty} [a_{n+1}(n+1) - a_n(n+1)] x^n = 0$$

$\Rightarrow$ $a_1 = a_0$,
$a_{n+1} = a_n$

$\Rightarrow$ $a_n = a_0$

So, $y(x) = \sum_{n=0}^{\infty} a_0 x^n$
$$= a_0 \sum_{n=0}^{\infty} x^n = \frac{a_0}{1-x}.$$

$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1$, conv. abs. for $|x| < 1$.

Ex. $y'' + 4xy' + 4y = 0$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 4\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + 4a_n] x^n + 4\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n$$
$$= (a_2(2)(1) + 4a_0) + \sum_{n=1}^{\infty} [a_{n+2}(n+2)(n+1) + 4a_n + 4a_n \cdot n] x^n = 0$$

$\Rightarrow$ $a_2 = \frac{-4a_0}{2}$, $a_{n+2} = \frac{-4(n+1)}{(n+2)(n+1)} a_n = \frac{-4}{(n+2)} a_n$

So, $\underline{n=0}$: $a_2 = \frac{-4}{2} \cdot a_0 = \frac{-4}{2} \cdot \frac{a_0}{1!}$ $\underline{n=1}$: $a_3 = \frac{-4}{3} \cdot a_1 = \frac{-4}{3 \cdot 2 \cdot 1} \cdot 2a_1$

$\underline{n=2}$: $a_4 = \frac{-4}{4} \cdot a_2 = \frac{-4}{4} \cdot \frac{-4}{2} a_0 = \frac{(-4)^2 a_0}{2^2 \cdot 2 \cdot 1}$ $\underline{n=3}$: $a_5 = \frac{-4}{5} a_3 = \frac{-4}{5} \cdot \frac{-4}{3} \cdot a_1 =$

$\underline{n=4}$: $a_6 = \frac{-4}{6} \cdot a_4 = \frac{-4 \cdot (-4)(-4)}{6 \cdot 4 \cdot 2} = \frac{(-4)^3 a_0}{2^3 \cdot 3 \cdot 2 \cdot 1}$ $\underline{n=5}$: $a_7 = \frac{-4}{7} \cdot a_5 = \frac{(-4)}{7} \cdot \frac{(-4)}{5} \cdot \frac{(-4)}{3} \cdot a_1$

So, $a_{2n} = \frac{(-4)^n}{2^n \cdot n!} a_0$, $a_{2n+1} = \frac{(-4)^n \cdot 2^n n!}{(2n+1)!} a_1$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$
$$= a_0 \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^{2n} + a_1 \sum_{n=0}^{\infty} (-8)^n \frac{n!}{(2n+1)!} x^{2n+1}$$

$(2n)!! = 2^n \cdot n!$

$(2n-1)!! = \frac{(2n-1)!}{2^{n-1}(n-1)!}$

$(2\gamma+1)!! = \frac{(2\gamma+1)!}{2^n \gamma!}$

$2n-1 = 2\gamma+1$

$n = \frac{2(\gamma+1)}{2} = \gamma+1$

$\underline{\text{R.o.C.}}$: $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-2)^n}{n!} \cdot \frac{(n+1)!}{(-2)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2} \right| = \infty$

and $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-8)^n \cdot n!}{(2n+1)!} \cdot \frac{(2(n+1)+1)!}{(n+1)! \cdot (-8)^{n+1}} \right|$

$$= \frac{1}{8} \lim_{n\to\infty} \frac{(2(n+1)+1) \cdot (2(n+1)) \cancel{(2(n+1)-1)!} \cdot \cancel{n!}}{\cancel{(2n+1)!}\,(n+1)\cancel{n!}}$$
$$= \frac{1}{8} \lim_{n\to\infty} \frac{(2(n+1)+1)(2\cancel{(n+1)})}{\cancel{(n+1)}} = \frac{1}{4} \lim_{n\to\infty} 2(n+1)+1 = \infty$$

$\Rightarrow$ converges everywhere!