MATH201 - 800 Lab 4

Friday, October 1, 2021 8:59 AM

- Undetermined Coefficients
- Variation of Parameters.

Ex. $y'' + y' - 2y = 4e^x$. $(=0)$

U.C.: Make a "good" guess. Maybe try $\boxed{y_p(x) = Ae^x}$.

Find hom. sol$^{\underline{n}}$ first! Char. Eq$^{\underline{n}}$: $\dfrac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1}$ $\leftarrow >0 \Rightarrow$ case I

$= \dfrac{-1 \pm 3}{2} = 1, -2$

So,

$y_h(x) = c_1e^x + c_2e^{-2x}$

Can't guess $y_p(x) = Ae^x$!

Need to multiply by x! Try instead: $\boxed{y_p = Axe^x}$.

Take derivatives, plug into the original eq$^{\underline{n}}$, solve for A:

$y_p' = Ae^x + Axe^x$

$y_p'' = 2Ae^x + Axe^x$

$\underbrace{y_p'' + y_p' - 2y_p}_{= 4e^x} = (2Ae^x + \cancel{Axe^x}) + (Ae^x + \cancel{Axe^x}) - 2(\cancel{Axe^x})$

$= 3Ae^x \Rightarrow A = \dfrac{4}{3}$.

$y_p(x) = \dfrac{4}{3}xe^x$,

$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{-2x} + \dfrac{4}{3}xe^x$.

Ex. $2y'' + 4y' + 2y = e^{-x}$. Guess: Ae^{-x}.

Hom. Sol$^{\underline{n}}$: $\dfrac{-4 \pm \sqrt{16 - 4\cdot 2\cdot 2 = 0}}{2\cdot 2} = -1 \Rightarrow$ case II

$y_h(x) = \boxed{c_1e^{-x} + c_2xe^{-x}}$. Can't guess Ae^{-x}.

Try instead: Axe^{-x} ✗ Also won't work.

Try instead: $Ax^2e^{-x} = y_p(x)$.

Take der., sub into orig. eq$^{\underline{n}}$, solve for A!

$y_p' = 2Axe^{-x} - Ax^2e^{-x}$

$y_p'' = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$

$= Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x}$.

$2y_p'' + 4y_p' + 2y_p = 2(Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x})$
$+ 4(2Axe^{-x} - Ax^2e^{-x})$
$+ 2(Ax^2e^{-x})$
$= (\cancel{2A - 4A + 2A})^{0}\, x^2e^{-x}$
$+ (\cancel{-8A + 8A})^{0}\, xe^{-x}$
$+ 4Ae^{-x}$

$e^{-x} = 4Ae^{-x} \Rightarrow \boxed{A = \dfrac{1}{4}}$!

$y_p(x) = \dfrac{1}{4}x^2e^{-x}$, and

$y(x) = y_h(x) + y_p(x)$

$= c_1e^{-x} + c_2xe^{-x} + \dfrac{1}{4}x^2e^{-x}$.

V.o.P.: Given hom. sol$^{\underline{n}}$'s $y_1(x), y_2(x)$ solving

$y'' + P(x)y' + Q(x)y = R(x)$

Then $y_p(x) = u_1\cdot y_1 + u_2\cdot y_2$, where

$u_1 = -\displaystyle\int \frac{y_2\, R(x)}{W(y_1,y_2)}\,dx$,

$u_2 = \displaystyle\int \frac{y_1\cdot R(x)}{W(y_1,y_2)}\,dx$,

and $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$.

Ex. $2y'' + 4y' + 2y = e^{-x}$

↳ Std. Form: $y'' + 2y' + y = \frac{1}{2}e^{-x} = R(x)$.

$y_h(x) = c_1e^{-x} + c_2xe^{-x}$

0. $y_1(x) = e^{-x}$, $y_2(x) = xe^{-x}$,

1. $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$

$= e^{-x}(e^{-x} - \cancel{xe^{-x}}) + (\cancel{xe^{-x}})(e^{-x})$

$= e^{-2x} \neq 0$

2. Find u_1 & u_2: $u_1 = -\displaystyle\int \frac{x\cancel{e^{-x}}\cdot\frac{1}{2}\cancel{e^{-x}}}{\cancel{e^{-2x}}}$

$= -\dfrac{1}{2}\dfrac{x^2}{2} = \dfrac{-x^2}{4}$

$u_2 = \displaystyle\int \frac{\cancel{e^{-x}}\cdot\frac{1}{2}\cancel{e^{-x}}}{\cancel{e^{-2x}}} = \frac{1}{2}\int dx = \frac{x}{2}$.

3. $y_p(x) = y_1u_1 + y_2u_2$

$= e^{-x}\cdot\left(-\dfrac{x^2}{4}\right) + xe^{-x}\cdot\dfrac{x}{2}$

$= \dfrac{1}{4}x^2e^{-x}$

So, $y(x) = y_h(x) + y_p(x)$

$= c_1e^{-x} + c_2xe^{-x} + \dfrac{1}{4}x^2e^{-x}$

Ex. $y'' - 2y' + y = \dfrac{e^x}{x^2+1}$

Undetermined Coeff's:
- exponentials
- sin / cos
- polynomials
- combinations.

0. hom. sol$^{\underline{n}}$! $\dfrac{2 \pm \sqrt{4 - 4\cdot 1\cdot 1 = 0}}{2\cdot 1} = 1. \Rightarrow$ case II!

$y_h(x) = c_1e^x + c_2xe^x$

$y_1(x) = e^x$, $y_2(x) = xe^x$

1. $W(y_1,y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + \cancel{xe^x}) - \cancel{e^xxe^x}$
$= e^{2x} \neq 0$

2. Solve for u_1 & u_2!

$u_1 = -\displaystyle\int \frac{y_2\, R(x)}{W(y_1,y_2)}\,dx = -\int \frac{x\cancel{e^x}\cdot\cancel{e^x}}{(x^2+1)}\cdot\frac{1}{\cancel{e^{2x}}}\,dx$

$= -\displaystyle\int \frac{x}{x^2+1}\,dx$, sub $w = x^2+1$, $\dfrac{dw}{2} = 2x\,dx$

$= -\dfrac{1}{2}\displaystyle\int \frac{1}{w}\,dw = -\frac{1}{2}\ln(w)$

$u_1 = -\dfrac{1}{2}\ln(x^2+1)$

$u_2 = \displaystyle\int \frac{y_1 R}{W(y_1,y_2)}\,dx = \int \frac{\cancel{e^x}\cdot\cancel{e^x}}{(x^2+1)}\cdot\frac{1}{\cancel{e^{2x}}}\,dx$

$= \displaystyle\int \frac{1}{x^2+1} = \arctan(x)$.

$y_p(x) = y_1\cdot u_1 + y_2\cdot u_2$

$= e^x\cdot\left(-\dfrac{1}{2}\ln(x^2+1)\right) + xe^x\cdot\arctan(x)$.

So, $y(x) = y_h(x) + y_p(x)$

$= c_1e^x + c_2xe^x + e^x\left[x\arctan(x) - \dfrac{1}{2}\ln(x^2+1)\right]$

Ex. $ty'' + (t+1)y' + y = t^2$, where

$y_1(t) = e^t$,
$y_2(t) = t+1$, are hom. sol$^{\underline{n}}$'s!

1. Std. Form! $1\cdot y'' + \left(\dfrac{t+1}{t}\right)y' + \dfrac{1}{t}y = \underset{R(t)}{\underset{\shortparallel}{t}}$, $t > 0$.

2. $W(y_1,y_2) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix}$

$= e^t - (t+1)e^t = -te^t \neq 0$ for $t \neq 0$.

3. Solve for u_1 & u_2:

$u_1 = -\displaystyle\int \frac{y_2\, R(t)}{W(y_1,y_2)}\,dt$

$= \cancel{-}\displaystyle\int \frac{(t+1)\cdot\cancel{t}}{\cancel{-t}e^t} = \int (t+1)e^{-t}$

$= -e^{-t} + \displaystyle\int te^{-t}$, $u = t$, $du = dt$; $dv = e^{-t}$, $v = -e^{-t}$

$= -e^{-t} + (-te^{-t}) + \displaystyle\int e^{-t}\,dt$

$u_1 = -2e^{-t} - te^{-t} = -e^{-t}(t+2)$

$u_2 = \displaystyle\int \frac{e^t\cdot t}{-te^t} = -\int 1\,dt = -t$

$y_p(t) = y_1u_1 + y_2u_2$

$= \cancel{e^t}(-\cancel{e^{-t}}(t+2)) + (t+1)(-t)$

$= -(t^2 + 2t + 2)$

Final Sol$^{\underline{n}}$: $y(x) = y_h(t) + y_p(t)$

$y(x) = c_1e^t + c_2(t+1) - (t^2 + 2t + 2)$