MATH 201 - 800 lab 5

Friday, October 8, 2021 11:32 AM

- Reduction of order
- Mechanical Vibrations

Suppose we have an ODE of the form:

$$\underset{w'}{y''} = F(x, \underset{w}{y'})$$

set $w = y' \Rightarrow w' = y''$

Eq^{n} can be written: $\boxed{w' = F(x, w)}$

Ex. $2t^2y'' + 3ty' - y = 0$, $t > 0$,

given that $y_1(t) = t^{-1}$ is a hom. sol^{n}.

$y_1, y_2 \to y_0 = y_1 \cdot u_1 + y_2 \cdot u_2$

Reduction of order: set $y_2(t) = \boxed{y_1(t) \cdot v(t)}$

$$y_2' = y_1'v + y_1 v'$$

$$y_2'' = y_1''v + 2y_1'v' + y_1 v''$$

Plug in:

$$0 = 2t^2(y_1''v + 2y_1'v' + y_1 v'') + 3t(y_1'v + y_1 v') - (y_1 v)$$

$$0 = 2t^2 y_1 v'' + (4t^2 y_1' + 3t y_1)v' + \underbrace{(2t^2 y_1'' + 3ty_1' - y_1)}_{=0 \text{ since } y_1 \text{ is a sol}^{n}!}v$$

$y_1 = t^{-1}$

$y_1' = -t^{-2}$

$$0 = 2t^2 \cdot t^{-1} v'' + (4\cancel{t^2}(-\cancel{t^{-2}}) + 3\cancel{t}\cancel{t^{-1}})v'$$

$$= 2tv'' + (-4 + 3)v'$$

$$= 2tv'' - v'$$

Stnd. Form: $v'' = \frac{1}{2t}v' = F(t, v')$

sub. $w = v'$, $w' = v''$

New eq^{n}:

$$w' = \frac{1}{2t} \cdot w \rightarrow \text{separable!}$$

$$\int \frac{dw}{w} = \int \frac{dt}{2t}$$

$$\ln(w) = \frac{1}{2}\ln(t) = \ln(t^{1/2})$$

$$\boxed{w(t) = t^{1/2}}$$

Since $w = v'$, $\int w = \int v' = v$

So, $v(t) = \int t^{1/2} = \boxed{\frac{2}{3}t^{3/2}}$

So, $y_2(t) = y_1(t) \cdot v(t)$

$$= t^{-1} \cdot \frac{2}{3}t^{3/2} = \frac{2}{3}\boxed{t^{1/2}}$$

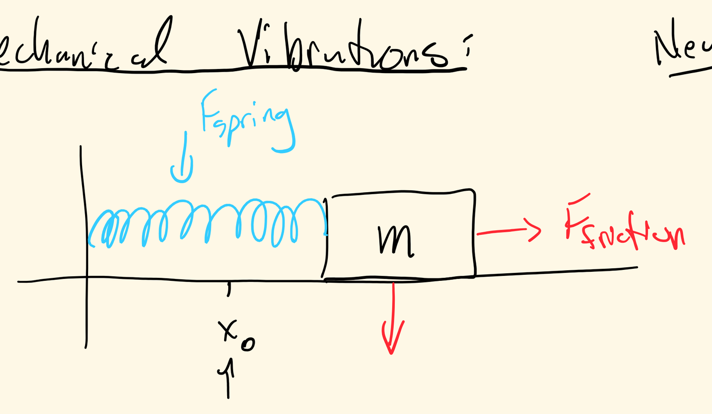
Hence, $y_1(t) = c_1 t^{-1}$

$y_2(t) = c_1 t^{1/2}$

and $y_h(x) = c_1 t^{-1} + c_2 t^{1/2}$.

Mechanical Vibrations:

Newton: $F = ma$



Denote position from x_0 by $y(t)$.
(when $y = 0$, box is at x_0)

Then, since $y(t)$ - position / displacement

$\frac{dy}{dt}$ - velocity

$\frac{d^2y}{dt^2}$ - acceleration

It turns out:

$F_{spring} \sim$ displacement

$F_{friction} \sim$ velocity

$$\Rightarrow m \cdot \frac{d^2y}{dt^2} = F$$

$$= -F_{spring} - F_{friction}$$

$$= -K \cdot y(t) - b \cdot \frac{dy}{dt}$$

So: $my'' + by' + Ky = 0$

a second order linear const. coeff. eq^{n}!

Always assume: $m > 0$, $K > 0$, $b \geq 0$

m - mass

K - spring constant

b - coefficient of fric.

When $b = 0 \Rightarrow$ "Free vibrations". $\rightarrow$ If $b = 0$, Char. Eq^{n} is:

$$\frac{0 \pm \sqrt{0 - 4mK}}{2m} = \pm\sqrt{\frac{-K}{m}} = \pm i\sqrt{\frac{K}{m}}.$$

Sol^{n} is: $\underbrace{c_1\cos(\sqrt{\tfrac{K}{m}}x) + c_2\sin(\sqrt{\tfrac{K}{m}}x)}_{\text{solutions oscillate forever!}}$

3 Cases for Mechanical Vibrations:

Case I: $b^2 - 4mK > 0 \Rightarrow$ over damped

Case II: $b^2 - 4mK = 0 \Rightarrow$ critically damped

Case III: $b^2 - 4mK < 0 \Rightarrow$ underdamped

Case I: two real, distinct roots: $\frac{-b \pm \sqrt{b^2 - 4mK}}{2m} < 0$

both NEGATIVE!

$\Rightarrow$ sol^{n} $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

and $\lim_{t \to \infty} y(t) = 0$