Lab One

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<u>Topics:</u>
- separable equations
- homogeneous equations
- equations of type $G(ax+by) = y'$

1. Suppose we are given a first order differential equation that looks like

$$y' = f(x,y), \quad \text{or}$$
$$\frac{dy}{dx} = f(x,y).$$

Then x is the independent variable, and $y = y(x)$ is the dependent variable we want to solve for.

If the equation above can be written as

$$\frac{dy}{dx} = g(x)h(y),$$

for some functions g, h, then the equation is called <u>separable</u>!

This name makes sense, as we were able to take the original function $f(x,y)$ and <u>separate</u> it into two pieces, $g(x)$ and $h(y)$.

<u>Examples:</u> (1) $$\frac{dy}{dx} = \frac{x^2}{1-y^2}.$$

This equation <u>is</u> separable. We have

$$f(x,y) = \frac{x^2}{1-y^2} = (x^2)\left(\frac{1}{1-y^2}\right).$$

So, in this case, $g(x) = x^2$,

$$h(y) = \frac{1}{1-y^2}.$$

(2) $$\frac{dy}{dx} = \frac{x^2 - x^3y}{(1-xy)(1+y)}$$

This equation <u>is also</u> separable, but it may not be as obvious as the previous example. This time we have

$$f(x,y) = \frac{x^2 - x^3y}{(1-xy)(1+y)}.$$

If we simplify the numerator, we see that

$$f(x,y) = \frac{x^2-x^3y}{(1-xy)(1+y)} = \frac{x^2\cancel{(1-xy)}}{\cancel{(1-xy)}(1+y)} = \frac{x^2}{1+y}.$$

So, $g(x) = x^2$ and $h(y) = \frac{1}{1+y}$.

(3) $$\frac{dy}{dx} = \frac{(x^2+y^2)}{1+y}$$

This equation <u>is not</u> separable, though I encourage you to give it a try just to see for yourself!

<u>Solving Separable Equations</u>

Since the equation can be separated, splitting up the independent and dependent variables, one can rewrite the equation as:

$$\frac{dy}{dx} = g(x)h(y), \quad \text{or}$$
$$\frac{dy}{h(y)} = g(x)\,dx.$$

One then integrates both sides and solves for $y(x)$ (if possible). Don't forget the constant!!

<u>Examples:</u> (1) $$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

<u>Step 1.</u> Separate! $(1-y^2)\,dy = x^2\,dx$

<u>Step 2.</u> Integrate!

$$\int (1-y^2)\,dy = \int x^2\,dx$$
$$\Rightarrow \quad y - \frac{y^3}{3} = \frac{x^3}{3} + C.$$

This is a solution that is <u>implicit</u>. We have our solution, but we <u>cannot</u> write y as an <u>explicit</u> function of x.

(2) $$\frac{dy}{dx} = \frac{x^2}{y}.$$

<u>Step 1.</u> Separate! $y\,dy = x^2\,dx$.

<u>Step 2.</u> Integrate!

$$\int y\,dy = \int x^2\,dx$$
$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^3}{3} + C$$
$$\Rightarrow \quad y^2 = \frac{2x^3}{3} + 2C$$
$$\Rightarrow \quad y(x) = \pm\sqrt{\frac{2x^3}{3} + 2C}$$

In this example, we have an <u>explicit</u> solution.

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2. A <u>homogeneous</u> equation is an equation that can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

That is, everything on the right hand side can be written as a ratio of y/x.

<u>Examples:</u> (1) $$\frac{dy}{dx} = \frac{y-4x}{x-y}.$$

Our goal is to rewrite the equation in ratios of $\frac{y}{x}$.

For the numerator, we can instead write

$$y - 4x = x\left(\frac{y}{x} - 4\right)$$

For the denominator, we can instead write

$$x - y = x\left(1 - \frac{y}{x}\right).$$

Combining this allows us to write

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\cancel{x}\left(\frac{y}{x}-4\right)}{\cancel{x}\left(1-\frac{y}{x}\right)} = \frac{y/x - 4}{1 - y/x}.$$

Everything is now in terms of y/x, and so the equation is indeed homogeneous.

<u>Solving Homogeneous Equations</u>

Now that we know the equation can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

we make the substitution $u := \frac{y}{x}$.

Then,

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$
$$= \frac{1}{x}\left(\frac{dy}{dx} - u\right)$$
$$= \frac{1}{x}\left(f(u) - u\right).$$

Hence, $x\frac{du}{dx} = f(u) - u$, which is separable!

<u>Example:</u>

$$\frac{dy}{dx} = \frac{y-4x}{x-y}. \quad \text{From our previous work,}$$
$$\frac{dy}{dx} = \frac{y/x - 4}{1 - y/x}. \quad \text{With the substitution } u = y/x,$$
$$\frac{dy}{dx} = \frac{u-4}{1-u} =: f(u).$$

Now we can write

$$x\frac{du}{dx} = f(u) - u$$
$$= \frac{u-4}{1-u} - u$$
$$= \frac{u - 4 - u(1-u)}{1-u}$$
$$= \frac{u^2-4}{1-u}.$$

We can now solve this as it is separable!

$$x\frac{du}{dx} = \frac{u^2-4}{1-u} \Rightarrow \frac{(1-u)}{u^2-4}\,du = x\,dx$$
$$\Rightarrow \quad \int \frac{1-u}{u^2-4}\,du = \int x\,dx = \frac{x^2}{2} + C.$$

In order to integrate the left hand side, recall that

$$\int \frac{1}{u^2-4} = \int \frac{1}{(u+2)(u-2)} = \frac{1}{4}\int\left(\frac{1}{u-2} - \frac{1}{u+2}\right)du$$
$$= \frac{1}{4}\left(\ln(u-2) - \ln(u+2)\right) + C.$$

Also, $-\int \frac{u}{u^2-4}$ can be solved via substitution: $v := u^2-4$, $\frac{dv}{2} = u\,du$,

and so

$$-\int \frac{u}{u^2-4} = -\frac{1}{2}\int \frac{dv}{v}$$
$$= -\frac{1}{2}\ln(v)$$
$$= -\frac{1}{2}\ln(u^2-4).$$

To conclude,

$$\int \frac{1-u}{u^2-4} = \frac{1}{4}\left[\ln(u-2) - \ln(u+2)\right] - \frac{1}{2}\ln(u^2-4).$$

Our (implicit) solution is then:

$$\frac{1}{4}\left[\ln(u-2) - \ln(u+2)\right] - \frac{1}{2}\ln(u^2-4) = \frac{x^2}{2} + C.$$

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3. Sometimes, an equation can be written in the form

$$\frac{dy}{dx} = f(ax+by),$$

for some constants a and b.

Similar to the homogeneous equations, we can make the substitution $u = ax + by$, and so

$$\frac{du}{dx} = \frac{d}{dx}(ax + by)$$
$$= a + b\frac{dy}{dx}.$$

Thus,

$$\frac{du}{dx} = a + b\cdot f(u),$$

which is a separable equation!

<u>Example:</u>

$$\frac{dy}{dx} = (x-y)^2.$$

Here, $f(ax+by) = (ax+by)^2$, and so $a := 1$, $b := -1$.

We can make the substitution $u = x - y$, and our equation becomes

$$\frac{du}{dx} = 1 - u^2.$$

This is separable, so we write this as

$$\frac{du}{1-u^2} = dx,$$

and solve as we are now used to.

In this case,

$$\ln\left[\sqrt{\frac{u+1}{u-1}}\right] = x + C,$$

or

$$\sqrt{\frac{u+1}{u-1}} = Ae^x \quad (\text{set } A := e^C)$$

or

$$\frac{u+1}{u-1} = A^2e^{2x}$$

or

$$u = \frac{A^2e^{2x}+1}{A^2e^{2x}-1}$$

From our original substitution, $u = x - y$, and so

$$y = x - \frac{A^2e^{2x}+1}{A^2e^{2x}-1}$$