MATH201 - 800 Lab 2

Friday, September 17, 2021 8:54 AM

Today:
- Linear Equations
- Bernoulli Equations
- Exact Equations

Linear Eq^{n}'s: $1\cdot\frac{dy}{dx} + P(x)y = Q(x)$ Stnd. Frm.

To solve: $\mu(x) = e^{\int P(x)dx}$

$$\Rightarrow \quad y(x) = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx.$$

Ex. $x\frac{dy}{dx} + x^2y = xe^x, \quad x>0.$

Step 1. Stnd. Frm. $\frac{dy}{dx} + \underbrace{x}_{P(x)}y = \underbrace{e^x}_{Q(x)}.$

Step 2. Int. Fact. $\mu(x) = e^{\int x} = e^{x^2/2}.$

$$\underbrace{e^{x^2/2}\frac{dy}{dx} + xe^{x^2/2}y}_{} = e^{x^2/2+x}$$

$$\int\frac{d}{dx}\left(e^{x^2/2}\cdot y(x)\right)dx = \int e^{x^2/2+x}dx$$

$$e^{x^2/2}y(x) = \int e^{x^2/2+x}dx$$

$$y(x) = e^{-x^2/2}\int e^{x^2/2+x}dx$$

$$= \frac{1}{\mu(x)}\int \mu(x)Q(x)dx$$

Ex. $x\frac{dy}{dx} + 2y = x^2.$

Step 1. Stnd. Frm. $\frac{dy}{dx} + \frac{2}{x}y = x$

Step 2. Int. Fact. $\mu(x) = e^{\int\frac{2}{x}dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$

Step 3. Solve

$$\underbrace{x^2\frac{dy}{dx} + 2xy}_{} = x^3$$

$$\int\frac{d}{dx}(x^2\cdot y) = \int x^3$$

$$x^2y(x) = \frac{x^4}{4} + C$$

$$y(x) = \frac{x^2}{4} + \frac{C}{x^2}.$$

Bernoulli Eq^{n}'s $y' + P(x)y = Q(x)y^n, \quad n\in\mathbb{R}.$ $n\neq 0, n\neq 1$

To solve: set $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$= (1-n)y^{-n}\left[Q(x)y^n - P(x)y\right]$$

$$= (1-n)\left[Q(x) - P(x)y^{1-n}\right]$$

$$\frac{du}{dx} = (1-n)\left[Q(x) - P(x)u\right]$$

or: $$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Ex. $t^2\frac{dy}{dt} + 2ty = y^3$

Step 1 Stnd. Frm. $\frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3$

$P(t) = \frac{2}{t}, \quad Q(t) = \frac{1}{t^2}, \quad n = 3$

Step 2. Convert to 1st order linear eq^{n}:

$$\frac{du}{dx} + (1-3)\cdot\frac{2}{t}u = (1-3)\cdot\frac{1}{t^2}$$

$$\frac{du}{dx} \underbrace{-\frac{4}{t}}_{R(t)}u = \underbrace{-\frac{2}{t^2}}_{S(t)}$$

Step 3. Solve using int. fact. $\mu(t) = e^{\int R(t)dt} = e^{-4\int\frac{1}{t}dt} = e^{-4\ln(t)} = t^{-4}$

$$\Rightarrow \quad u(t) = \frac{1}{\mu(t)}\int\mu(t)S(t)dt$$

$$= \frac{1}{t^{-4}}\int t^{-4}\cdot\left(\frac{-2}{t^2}\right)dt$$

$$= t^4\int -2t^{-6}dt$$

$$= t^4\left[-2\cdot\left(\frac{-1}{5}\cdot t^{-5}\right) + C\right]$$

$$u(t) = \frac{2}{5}t^{-1} + Ct^4$$

Step 4. Convert back to $y(t)$! $u = y^{1-n} \Rightarrow y = u^{\frac{1}{1-n}}$

$$y(t) = \left(\frac{2}{5}t^{-1} + Ct^4\right)^{\frac{1}{1-n}}$$

$$y(t) = \left(\frac{2}{5}t^{-1} + Ct^4\right)^{-\frac{1}{2}}.$$

$$y - y^3 = xe^x$$

$$y - x^2 = Ae^xy$$

$$y(1-Ae^x) = x^2$$

$$y(x) = \frac{x^2}{1-Ae^x}$$

Exact Eq^{n}'s: Suppose we have an eq^{n} of the form

$$M(x,y)dx + N(x,y)dy = 0$$

If $M_y = N_x$, it is called exact!

$$\frac{d}{dx}\Psi(x,y(x)) = \frac{d}{dx}C$$

$$\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\underbrace{\frac{\partial\Psi}{\partial x}}_{M}dx + \underbrace{\frac{\partial\Psi}{\partial y}}_{N}dy = 0$$

Also: $\Psi_{xy} = \Psi_{yx}$

$$\frac{\partial}{\partial y}\left(\frac{\partial\Psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial y}\right)$$

$$\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)$$

$$\begin{cases}\frac{\partial\Psi}{\partial x} = M\\ \frac{\partial\Psi}{\partial y} = N\end{cases}$$

Ex. $\frac{dy}{dx} = \frac{-(y\cos(x) + 2xe^y)}{(\sin(x) + x^2e^y - 1)}$

Rewrite: $\underbrace{(y\cos(x) + 2xe^y)}_{M(x,y)}dx + \underbrace{(\sin(x) + x^2e^y - 1)}_{N(x,y)}dy = 0$

$M_y = \cos(x) + 2xe^y$
$N_x = \cos(x) + 2xe^y$ > same $\Rightarrow$ EXACT!

Ex. $\underbrace{(3xy + y^2)}_{M}dx + \underbrace{(x^2 + xy)}_{N}dy = 0$

$M_y = 3x + 2y$
$N_x = 2x + y$ > NOT EQUAL! Not exact!

Solving Exact Eq^{n}'s: Use: $\begin{cases}\Psi_x = M\\ \Psi_y = N\end{cases}$

Ex. $(2x+3)dx + (2y-2)dy = 0$

$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$ is exact!

Eq^{n}'s: $\begin{cases}\Psi_x = 2x+3 & (I)\\ \Psi_y = 2y-2 & (II)\end{cases}$

(I) $\int\Psi_x dx = \Psi(x,y) = \int(2x+3)dx = x^2 + 3x + c(y)$

$$\Psi(x,y) = x^2 + 3x + c(y) \leftarrow$$

(II)

$$\Psi_y = 2y - 2$$

$$\frac{\partial\Psi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3x + c(y)\right) = c'(y)$$

$$\int c'(y) = \int 2y - 2$$

$$c(y) = y^2 - 2y.$$

So, $\Psi(x,y) = x^2 + 3x + y^2 - 2y$

And sol^{n} is: $$x^2 + 3x + y^2 - 2y = C$$

Ex. $(y\cos(x) + 2xe^y)dx + (\sin(x) + x^2e^y - 1)dy = 0$

Step 1. Check for exactness (done already!)

Step 2. Set up eq^{n}'s:

$$\begin{cases}\Psi_x = M = y\cos(x) + 2xe^y\\ \Psi_y = N = \sin(x) + x^2e^y - 1\end{cases}$$

$$\int\Psi_x dx = \Psi(x,y) = \int M dx$$

$$= \int(y\cos(x) + 2xe^y)dx$$

$$\Psi = (y\sin(x) + x^2e^y + c(y))$$

$$\Psi_y = \sin(x) + x^2e^y + c'(y)$$

$$N = \sin(x) + x^2e^y - 1$$

So: $\sin(x) + x^2e^y + c'(y) = \sin(x) + x^2e^y - 1$

$$\int c'(y) = \int -1$$

$$c(y) = -y$$

Step 3. Put it together $\Psi(x,y) = y\sin(x) + x^2e^y - y$

Final Sol^{n}: $$y\sin(x) + x^2e^y - y = C$$