MATH201 - 800 Lab One

Friday, September 10, 2021 10:58 AM

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Morning section for MATH201 (800)
Quiz Schedule is on eClass.
Notes are also on eClass.

9 - 10 am MST
5 - 6 pm MST

- Separable eq='s
- Homogeneous eq='s
- Eq='s of the form $y' = G(ax+by)$

Suppose you have an eq= of the form

$$\frac{dy}{dx} = y' = F(x,y) = g(x)\cdot h(y)$$

then the eq= is called separable.

Ex. (1) $\frac{dy}{dx} = \frac{x^2}{1-y^2} = (x^2)\cdot\left(\frac{1}{1-y^2}\right)$, with $g(x) = x^2$, $h(y) = \frac{1}{1-y^2}$. separable!

(2) $\frac{dy}{dx} = \frac{x^2 - x^3 y}{(1-xy)(1+y)}$

$= \frac{x^2(1-xy)}{(1-xy)(1+y)} = (x^2)\cdot\left(\frac{1}{1+y}\right)$ separable!

(3) $\frac{dy}{dx} = \frac{(x^2+y^2)}{1+y}$ Not separable!

Solving Sep. Eq='s: $\frac{dy}{dx} = g(x)\,h(y)$

Step 1. Rewrite: $\frac{dy}{h(y)} = g(x)dx$

Step 2. Integrate: $\int \frac{1}{h(y)}dy = \int g(x)dx$.

Ex. (1) $\frac{dy}{dx} = x^2 \cdot \frac{1}{1-y^2}$

Step 1. Rewrite: $(1-y^2)\,dy = x^2\,dx$

Step 2. Integrate: $\int(1-y^2)\,dy = y - y^3/3 + c$

$\int x^2\,dx = \frac{x^3}{3} + d$

$\Rightarrow \boxed{y - \frac{y^3}{3} = \frac{x^3}{3} + C}$ → Implicit Sol=! (Can't write as $y(x) = \sim$)

(2) $\frac{dy}{dx} = \frac{x^2}{y}$.

Step 1. Rewrite: $y\,dy = x^2\,dx$

Step 2. Integrate: $\int y\,dy = \frac{y^2}{2}$, $\int x^2 dx = \frac{x^3}{3}$

$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C$

Step 3. Solve for $y(x)$: $y^2 = \frac{2}{3}x^3 + 2c$

$\boxed{y(x) = \pm\sqrt{\frac{2}{3}x^3 + 2c}}$

Explicit!

Homogeneous Eq='s: A hom. eq= is of the form:

$$\boxed{\frac{dy}{dx} = F\left(\frac{y}{x}\right)}$$

Ex. (1) $\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{1}{x}(y-4x)}{\frac{1}{x}(x-y)}$

$= \frac{y/x - 4}{1 - y/x} = f\left(\frac{y}{x}\right)$ homogeneous!

Solving hom. eq='s: Using the substitution $u = \frac{y}{x}$ we can turn this problem into a separable eq=!

So: $u = \frac{y}{x}$, $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

$\frac{du}{dx} = \frac{dy}{dx}\cdot\frac{1}{x} - \frac{1}{x^2}\cdot y$

$= \frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right)$

$= \frac{1}{x}\left(F\left(\frac{y}{x}\right) - \frac{y}{x}\right)$

$\boxed{\frac{du}{dx} = \frac{1}{x}\left(F(u) - u\right)}$ separable!

Ex. (1) $\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{y/x - 4}{1 - y/x} = f\left(\frac{y}{x}\right)$

Solve instead: $\frac{du}{dx} = \frac{1}{x}\left(f(u) - u\right)$

$= \frac{1}{x}\left(\frac{u-4}{1-u} - u\right)$

$= \frac{1}{x}\left(\frac{u - 4 - u(1-u)}{1-u}\right)$

$\frac{du}{dx} = \frac{1}{x}\left(\frac{u^2-4}{1-u}\right)$ $u = \frac{y}{x}$

Step 1. Rewrite: $\frac{(1-u)}{(u^2-4)}du = \frac{dx}{x}$

Step 2. Integrate: $\int\frac{dx}{x} = \boxed{\ln(x) + C}$

$\int\frac{(1-u)}{(u^2-4)}du = \underbrace{\int\frac{1}{u^2-4}du}_{\text{I}} - \underbrace{\int\frac{u}{u^2-4}du}_{\text{II}}$

(II): $-\int\frac{u}{u^2-4}du$, set $w = u^2-4$, $\frac{dw}{2} = u\,du$

$= -\frac{1}{2}\int\frac{dw}{w} = -\frac{1}{2}\ln(w) = \boxed{-\frac{1}{2}\ln(u^2-4)}$

(I) $\int\frac{1}{u^2-4}du = \int\frac{1}{(u+2)(u-2)}du$

$= \frac{1}{4}\int\left(\frac{1}{u-2} - \frac{1}{u+2}\right)du$

$= \boxed{\frac{1}{4}\left(\ln(u-2) - \ln(u+2)\right)}$

Put it all together: $\boxed{\frac{1}{4}\left(\ln(u-2) - \ln(u+2)\right) - \frac{1}{2}\ln(u^2-4) = \ln(x) + C}$

Implicit!

$\boxed{\frac{1}{4}\left(\ln\left(\frac{y}{x}-2\right) - \ln\left(\frac{y}{x}+2\right)\right) - \frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2 - 4\right) = \ln(x) + C}$

Sometimes an eq= can be written as:

$y' = G(ax+by)$, where a, b are constants.

Substitute $u = ax+by$,

$\frac{du}{dx} = a + b\frac{dy}{dx}$

$= a + b\cdot G(ax+by)$

$\boxed{\frac{du}{dx} = a + b\cdot G(u)}$ separable!

Ex. $\frac{dy}{dx} = (x-y)^2 = f(ax+by)$,

where $a = 1$, $b = -1$, $f(u) = u^2$

Solve instead: $\frac{du}{dx} = 1 - 1\cdot f(u)$

$\boxed{\frac{du}{dx} = 1 - u^2}$ separable!

Step 1. Rewrite: $\frac{du}{1-u^2} = dx$

Step 2. Integrate: $\int dx = x + C$

$\int\frac{1}{1-u^2}du = \int\frac{1}{(1-u)(1+u)}du$, partial frac.

$= \frac{1}{2}\left(\ln(u+1) - \ln(u-1)\right)$

$= \frac{1}{2}\ln\left(\frac{u+1}{u-1}\right)$

$= \ln\left(\sqrt{\frac{u+1}{u-1}}\right)$

Put this together: $\boxed{\ln\left(\sqrt{\frac{u+1}{u-1}}\right) = x + C}$

↳ $\sqrt{\frac{u+1}{u-1}} = Ae^x$ (set $e^c = A$)

↳ $\frac{u+1}{u-1} = A^2e^{2x}$

↳ $u+1 = (u-1)A^2e^{2x}$

↳ $u(1 - A^2e^{2x}) = -(1 + A^2e^{2x})$

↳ $\boxed{u = \frac{1 + A^2e^{2x}}{A^2e^{2x} - 1}}$

Then: our original sub. $u = x - y$, so

$x - y = \frac{A^2e^{2x}+1}{A^2e^{2x}-1}$

↳ $\boxed{y(x) = x - \frac{A^2e^{2x}+1}{A^2e^{2x}-1}}$