Lab Ten

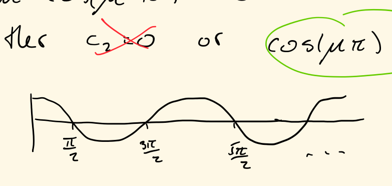
Wednesday, March 31, 2021 1:47 PM

§ 10.1, 10.2 - Eigenvalue Problems
- Fourier series.

Ex. $y'' + \lambda y = 0, \quad y(0) = y'(\pi) = 0$

$\underline{\lambda > 0}$: set $\lambda = \mu^2$. Then, $y'' + \mu^2 y = 0$
$\Rightarrow r^2 + \mu^2 = 0$
$\Rightarrow r = \pm i\mu.$

So, $y(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$
$y(0) = c_1 = 0 \Rightarrow c_1 = 0 \implies y(x) = c_2\sin(\mu x)$
$y'(x) = c_2\mu\cos(\mu x)$,
$y'(\pi) = c_2\mu\cdot\cos(\mu\cdot\pi) = 0$
$\Rightarrow$ either ~~$c_2 = 0$~~ or $\cos(\mu\pi) = 0$

Picture of cosine: 

So, $\cos(q) = 0$ when $q = \frac{(2k-1)\pi}{2}, \; k \in \mathbb{Z}$.

So, $\cos(\mu\pi) = 0$ when $\mu\cdot\pi = \frac{(2k-1)\pi}{2}$
$\hookrightarrow \mu = \frac{(2k-1)}{2}$

So, $\boxed{\lambda_n = \frac{(2k-1)^2}{4}, \quad k \in \mathbb{N}}$

$\underline{\lambda = 0}$: $y'' = 0 \Rightarrow y' = b$
$\Rightarrow y = bx + c, \quad y(0) = c = 0 \Rightarrow c = 0$
$y'(\pi) = b = 0 \Rightarrow b = 0$

Only the trivial solution in this case.

$\underline{\lambda < 0}$: set $\lambda = -\mu^2$. Then, $y'' - \mu^2 y = 0$
$\Rightarrow r^2 = \mu^2$
$\Rightarrow r = \pm\mu.$

So, $y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$
$y(0) = c_1 + c_2 \rightsquigarrow c_1 = -c_2$
$y'(x) = \mu(c_1 e^{\mu x} - c_2 e^{-\mu x})$,
$y'(\pi) = \mu(c_1 e^{\mu\pi} - c_2 e^{-\mu\pi})$
$\hookrightarrow y'(\pi) = \mu c_1(e^{\mu\pi} + e^{-\mu\pi}) \neq 0$ ever!

So, $c_1 = 0$ and hence $c_2 = 0$.
Only the trivial sol$^{\underline{n}}$ when $\lambda < 0$.

So, the <u>eigenvalues</u> are $\lambda_k = \frac{(2k-1)^2}{4}$ and
the <u>eigenfunctions</u> are $y_k(x) = \sin(\lambda_k x)$
$= \sin\left(\frac{(2k-1)^2 x}{4}\right)$

Ex. $y'' - \lambda y = 0, \quad y(0) = 0, \; y'(L) = 0$

$\underline{\lambda = 0}$: $y(x) = bx + c, \quad y(0) = c = 0 \Rightarrow c = 0$
$y'(L) = b = 0 \Rightarrow b = 0.$

Only trivial sol$^{\underline{n}}$!

$\underline{\lambda > 0}$: set $\lambda = \mu^2$, then $y'' - \mu^2 y = 0$
$\Rightarrow r = \pm\mu$
$\Rightarrow y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$

Repeat the previous steps to find $c_1 = c_2 = 0$
$\Rightarrow$ only the trivial sol$^{\underline{n}}$

$\underline{\lambda < 0}$: Set $\lambda = -\mu^2$. Then, $y'' - (-\mu^2)y = y'' + \mu^2 y = 0$
$\Rightarrow r = \pm i\mu.$

So, $y(x) = c_1\cos(\mu x) + c_2\sin(\mu x).$
$y(0) = c_1 = 0 \Rightarrow c_1 = 0.$
$y'(L) = c_2\mu\cos(\mu\cdot L) = 0$ (we want $c_2 \neq 0$)

So, $\cos(q) = 0$ when $q = \frac{(2k-1)}{2}\pi$, and so
$\cos(\mu\cdot L) = 0$ when $\mu\cdot L = \frac{(2k-1)}{2}\pi$
$\Rightarrow \mu = \frac{(2k-1)}{2L}\pi$

Hence, our eigenvalues are $\mu_k = \frac{(2k-1)}{2L}\pi$
and our eigenfunctions are
$y_k(x) = \sin(\mu_k x)$
$= \sin\left(\frac{(2k-1)\pi x}{2L}\right)$

So, what's the main idea / strategy here? First, we have an equation, some boundary conditions and a free parameter λ.

Using our knowledge for 2nd order ODE's, we first find a general solution. Using the boundary conditions, we then look for specific values of λ that solve the problem in a non-trivial way (i.e. $y(x) = 0$ is boring!)

There may be 0 (no sol$^{\underline{n}}$'s)
1 (a unique sol$^{\underline{n}}$)
or many solutions.

Once we find the eigenvalues λ_k, there is a solution which corresponds to each value. These are the eigenfunctions.

<u>Outline</u>:
1. Given an equation with a parameter λ, break into 3 cases: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.
2. Solve the problem for each case.
3. Use boundary conditions to solve for λ. (there may be many values of λ, or there may be none).
4. Once you have the eigenvalues, you can write down the eigenfunctions.

<u>Fourier Series</u>: Given a periodic function $f(x)$ on $[-L, L]$, we can write $f(x)$ as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

where

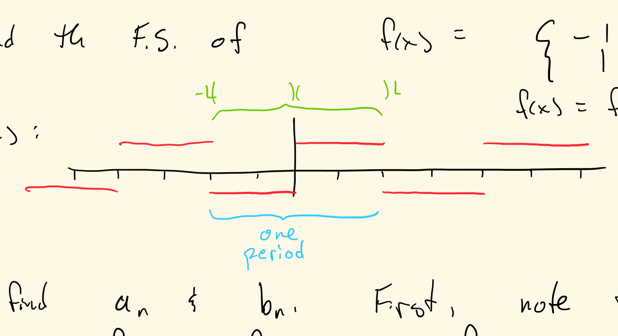
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

It's like magic!

Ex. Find the F.S. of $f(x) = \begin{cases} -1 & ; -2 \le x < 0 \\ 1 & ; 0 \le x < 2 \end{cases}$;
$f(x) = f(x+4)$

f(x): 

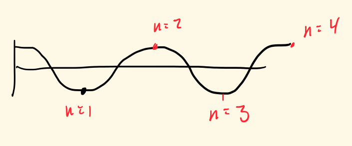
$\Rightarrow L = 2$

So, find a_n & b_n. First, note that
$f(x) = -f(-x) \Rightarrow f$ is ODD!

Then, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$ (f is odd!)

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = 0$ (cos is even, $f\cdot\cos$ = odd!)

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 2\cdot\frac{1}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ since $f\cdot\sin$ is even.

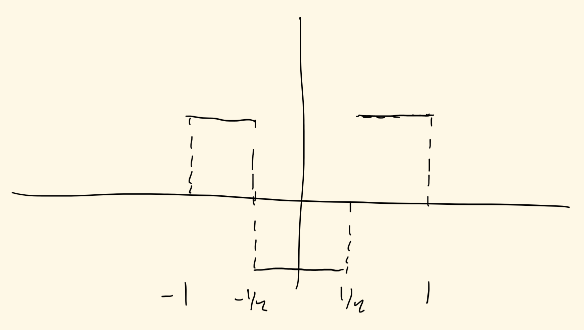
$b_n = 2\cdot\frac{1}{2}\int_0^2 1\cdot\sin\left(\frac{n\pi x}{2}\right)dx = 1\cdot\left(-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right)\Big|_0^2$
$= \frac{2}{n\pi}\left(-\cos(n\pi) + 1\right)$ $\rightsquigarrow$ 

$= \frac{2}{n\pi}\left(-(-1)^n + 1\right)$
$= \begin{cases} 4/n\pi & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$

So,

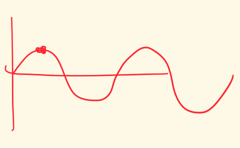
$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{2}\right)$$
$$= \sum_{k=1}^{\infty} b_{2k-1}\sin\left(\frac{(2k-1)\pi x}{2}\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi x/2)}{2k-1}$$

Ex. Find F.S. of $f(x) = \begin{cases} 1, & -1 \le x \le -1/2 \\ -1, & 1/2 < x \le 1/2 \\ 1, & 1/2 < x \le 1 \end{cases}$,
$f(x) = f(x+2)$

<u>Picture</u>: 

$f(x) = f(-x) \Rightarrow$ only $\cos(x)$ terms!

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 f(x)\cos(n\pi x)$$
$$= 2\left[\int_0^{1/2} -\cos(n\pi x) + \int_{1/2}^{1}\cos(n\pi x)\right]$$
$$= 2\left[-\frac{1}{n\pi}\sin(n\pi x)\Big|_0^{1/2} + \frac{1}{n\pi}\sin(n\pi x)\Big|_{1/2}^{1}\right]$$
$$= 2\left[-\frac{1}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - 0\right) + \frac{1}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$\sin\left(\frac{n\pi}{2}\right) \to (-1)^{n+1}$, $\sin(n\pi) \to 0$, $-\sin\left(\frac{n\pi}{2}\right) \to -(-1)^{n+1}$ 

$$= \frac{2}{n\pi}\left[-2(-1)^{n+1}\right] = \frac{4(-1)^n}{n\pi}$$

$a_0 = 0$

$$\Rightarrow f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(n\pi x)$$