Lab Nine

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Topics:
- Series solutions about ordinary points
- Radius of convergence

Power Series: First, it will be useful to recall what a power series is:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n .$$

Here, $\{a_n\}_{n=0}^{\infty}$ are the coefficients, x_0 is the centre, and all together we call this a power series centred at x_0.

Of coure, this series may not always converge for all $x \in (-\infty, \infty)$. But, usually there is sone region where the series is valid. We measure the size of this region by the radius of convergence.

Def^n: For each power series of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, there exists a number $0 \le \rho \le \infty$ such that for all x satisfying $|x-x_0| < \rho$, the series converges. If $|x-x_0| > \rho$, the series diverges. This value ρ is called the radius of convergence (or the R.o.C.).

Ratio Test: We can approximate the value ρ by considering the following limit:

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = L , \quad 0 \le L \le \infty .$$

If this limit exists, the value L is our radius of convergence.

Examples: Consider two power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} (x-2)^n$$

Compute the R.o.C. of each.

For the first, the centre is $x_0 = 0$. The coefficients are:

$$a_n = \frac{1}{n!},$$

So,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!}{n!}\right| \\ &= \lim_{n\to\infty}(n+1) = \infty. \end{aligned}$$

So, the R.o.C. is $\rho = \infty$! i.e. the series converges for any x!

For the second, the centre is $x_0 = 2$, and the coefficients are $a_n = 1$, So,

$$\begin{aligned} L = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty} 1 \\ &= 1. \end{aligned}$$

So, the R.o.C. is $\rho = 1$, and the series converges for all $|x-2| < 1$

$$\Rightarrow \quad -1 < x-2 < 1$$

$$\Rightarrow \quad \boxed{1 < x < 3}$$

Series Solutions to ODE's:

We can use properties of power series as a means to solve some ODE's. First, if we assume a series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a positive R.o.C. ρ, then

$$f(x) := \sum_{n=0}^{\infty} a_n(x-x_0)^n$$ is differentiable, and

$$\boxed{f'(x) = \sum_{n=1}^{\infty} a_n \cdot n(x-x_0)^{n-1}}, \quad \text{for } |x-x_0| < \rho.$$

Furthermore, the integral is given by:

$$\boxed{\begin{aligned}\int f(x)\,dx &= \int \sum_{n=0}^{\infty} a_n (x-x_0)^n \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{n+1}\end{aligned}}, \quad \text{for } |x-x_0| < \rho.$$

The main idea is to assume the existence of a power series solution to an ODE, plug it in and do some algebra to solve for the coefficients. So, suppose we have a general, second order ODE in standard form:

$$y'' + p(x)y' + q(x)y = 0.$$

Then, a point x_0 is called an ordinary point if both $p(x)$ and $q(x)$ are analytic at x_0. Otherwise, x_0 is called a singular point of the ODE.

So what does ANALYTIC mean? Essentially, analytic at a point x_0 means that the function has a power series representation centred at the point x_0.

Ex. Find all singular points of

$$x y'' + x(1-x)^{-1} y' + (\sin x) y = 0$$

Standard form: $y'' + (1-x)^{-1} y' + \frac{\sin x}{x} y = 0$

So, $p(x) = \frac{1}{1-x}$, $q(x) = \frac{\sin x}{x}$.

Since $\lim_{x\to 1} p(x) = \pm\infty$, $x = 1$ is a singular point.

For $q(x)$, recall that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$,

and so $\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n+1)!}$, and so $\frac{\sin x}{x}$ is actually analytic everywhere.

Hence, $x=1$ is the only singular point of the equation.

Now let's solve an equation using power series.

Ex. Solve

$$y' + 2xy = 0$$

about $x_0 = 0$.

So, assume a solution exists of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$= \sum_{n=0}^{\infty} a_n x^n.$$

Then plug this in. So,

$$y'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1},$$

$$2x \cdot y(x) = 2x \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

Then, the series must satisfy:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0.$$

In order to combine these, we need to combine like powers of x. This can be done in a number of ways, and I would suggest finding the way that makes the most sense to you.

First, I adjust the index on the first sum:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

and then I make the powers the same. So,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n} = a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^{n}$$

$$= a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1}$$

Now if we combine everything:

$$0 = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$= a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$= a_1 + \sum_{n=0}^{\infty}\left[(n+2)a_{n+2} + 2a_n\right]x^{n+1} ,$$

and in order for this to be zero for $x \neq 0$, we must have that

$$a_1 = 0 \quad \text{and} \quad (n+2)a_{n+2} + 2a_n = 0,$$

for all n.

From the second relation, we can solve for future a_n's knowing previous a_n's. That is,

$$a_{n+2} = -\frac{2}{n+2} \cdot a_n .$$

This is called a recurrence relation. So, let's plug in some values of n:

$n=0$: $a_2 = -\frac{2}{2}a_0 = -a_0$

$n=1$: $a_3 = -\frac{2}{3}a_1 = 0,$

$n=2$: $a_4 = \frac{-2}{4} \cdot a_2 = \frac{-1}{2} \cdot -a_0$

$n=3$: $a_5 = -\frac{2}{5} \cdot a_3 = 0$

$n=4$: $a_6 = -\frac{2}{6} \cdot a_4$

$$= \frac{-2}{6} \cdot \frac{-1}{2} \cdot -a_0$$

$$= \frac{-1}{3} \cdot \frac{-1}{2} \cdot -a_0$$

The idea is to look for a *pattern* so that we can write down a_n in general. First, we see that for any *odd* n, $a_n = 0$.
Then, for any *even* n, it seems to be:

$$\left(\frac{-1}{m}\right)\left(\frac{-1}{m-1}\right) \cdots\cdots \left(\frac{-1}{2}\right)(-1) \cdot a_0 \,,$$

but we need to figure out where m "starts".

First, when $n = 2$, we have 1

$n = 4$, we have $\frac{1}{2} \cdot \frac{1}{1}$

$n = 6$, we have $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$

$\vdots$

So, if $n = 2m$, we have $\frac{1}{m} \cdot \frac{1}{m-1} \cdots \frac{1}{2} \cdot \frac{1}{1}$.

In general, the formula for *even* n is:

$$a_n = (-1)^{n/2} \cdot \frac{1}{\left(\frac{n}{2}\right)!} \cdot a_0 .$$

Alternatively, if we write $n = 2m$, for any $m \in \mathbb{N}$, then

$$a_{2m} = (-1)^m \cdot \frac{1}{m!} a_0$$

Since $a_{2m+1} = 0$ for all n (since all odd indexed coefficients are zero), we can then write

$$a_m = (-1)^m \cdot \frac{1}{m!} \,.$$

Our final solution is then:

$$y(x) = \sum_{m=0}^{\infty} a_m \cdot (x - x_0)^{2m}$$

$$= a_0 \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{m!} \, x^{2m} .$$

This is a simple example, so one must practice to get a better idea of the tricks used to solve these problems!