Topics: - Fourier series! (And some tricks...)

Remember last time when we solved the following PDE:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

We found a series solution of the form:

$$u(x,t) = \sum_{n=0}^{\infty} e^{-(\frac{n\pi}{L})^2 t} c_n \sin(\tfrac{n\pi x}{L}).$$

What about those coefficients? Well, using the initial condition

$$u(x,0) = f(x),$$

we see that the following equality must hold:

$$u(x,0) = \sum_{n=0}^{\infty} c_n \sin(\tfrac{n\pi x}{L})$$
$$= f(x).$$

So.... if we were somehow able to choose the coefficients c_n so that

$$f(x) = \sum_{n=0}^{\infty} c_n \sin(\tfrac{n\pi x}{L}), \qquad (*)$$

we would have a true solution to our problem.

The crazy thing here is that we CAN choose our coefficients so that $(*)$ is true for a very large class of functions using FOURIER SERIES.

Some Useful Tricks

Before we go into Fourier series, let's review some tricks that will help us a *lot* with future calculations.

Recall that an EVEN function satisfies

$$f(x) = f(-x).$$

An ODD function satisfies

$$f(x) = -f(-x)$$

If we then integrate an EVEN function over a symmetric interval $(-a,a)$, we have:

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

If we integrate an ODD function over the interval $(-a,a)$, we have:

$$\int_{-a}^{a} f(x)dx = 0.$$

Ex. Find which functions are even, odd or neither.

1. $f(x) = \sqrt{1+x^2}$
2. $f(x) = x^{1/3} - \sin x$
3. $f(x) = e^x$

We can also look at the sum and product of even and odd functions.

- → the sum (or difference) of even functions is even
- → the sum (or difference) of odd functions is odd.
- → the product of two even functions is even
- → the product of two odd functions is even
- → the product of an odd function with an even function is an odd function
- → the same holds for quotients.

It is also useful to know that:

Even functions:
- $\cos x$
- x^n, where n is even
- $\cosh x$

Odd functions:
- $\sin x$
- x^n, where n is odd
- $\sinh x$

Fourier Series

We'll start with the definition.

Suppose $f(x)$ is piecewise continuous on $[-L,L]$. Then, the Fourier series of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(\tfrac{n\pi x}{L}) + b_n \sin(\tfrac{n\pi x}{L})\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{n\pi x}{L})dx,$$

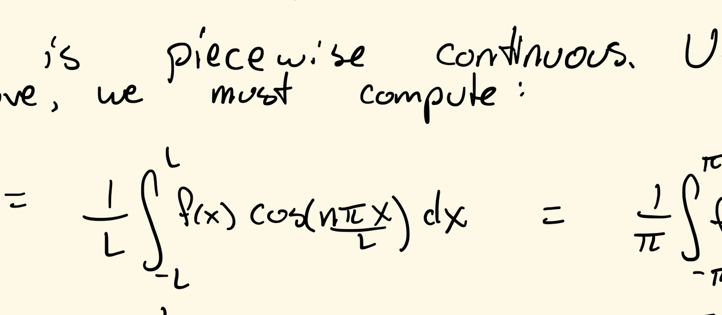
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{n\pi x}{L})dx.$$

Note: This series may not always converge, hence "~" written above. There are conditions that ensure this series *does* converge and is *equal* to $f(x)$, in some sense.

Ex. Compute the Fourier series for

$$f(x) := \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Here, $L = \pi$. $f(x)$ looks like:



So, f is piecewise continuous. Using our formula's above, we must compute:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{n\pi x}{L})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{n\pi x}{L})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

First, if we notice that $f(x)$ is odd, then

$$f(x)\cos(nx)$$

is odd for each $n = 0, 1, 2, \dots$

Hence,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$= 0$$

for each n.

Then, since $\sin(nx)$ is odd, $f(x)\sin(nx)$ is even, and so

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx,$$

but $f(x) = 1$ on $(0,\pi)$, and so

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx$$
$$= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi}$$
$$= \frac{2}{\pi}\left(-\frac{\cos(n\pi)}{n} + \frac{1}{n}\right).$$

Notice that $\cos(n\pi)$ oscillates between -1 and 1. In fact,

$$\cos(n\pi) = (-1)^n, \text{ and so}$$

$$b_n = \frac{2}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right)$$
$$= \begin{cases} \frac{4}{\pi}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

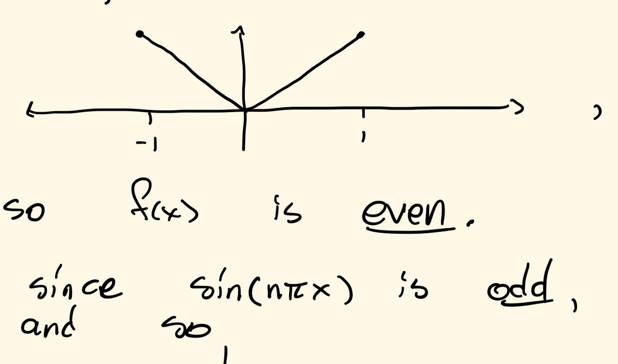
Thus, the Fourier series is:

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right)\sin(nx).$$

Ex. Compute the Fourier series of

$$f(x) = |x|, \quad -1 < x < 1.$$

This time, $f(x)$ looks like



and so $f(x)$ is even.

So, since $\sin(n\pi x)$ is odd, $f(x)\sin(n\pi x)$ is odd, and so

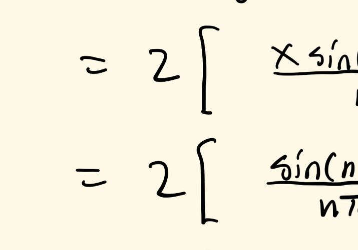
$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx = 0.$$

Then, since $\cos(n\pi x)$ is even, $f(x)\cos(n\pi x)$ is even, and so

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx$$
$$= 2\int_0^1 f(x)\cos(n\pi x)dx,$$

but $f(x) = |x| = x$ on $(0,1)$, and so

$$a_n = 2\int_0^1 x\cos(n\pi x)dx, \qquad u = x,\ du = 1,\quad dv = \cos(n\pi x),\ v = \frac{\sin(n\pi x)}{n\pi}$$
$$= 2\left[\frac{x\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\right]$$
$$= 2\left[\frac{\sin(n\pi)}{n\pi} - \frac{1}{n\pi}\left(-\frac{\cos(n\pi x)}{n\pi}\Big|_0^1\right)\right]$$
$$= 2\left[\frac{\sin(n\pi)}{n\pi} - \frac{1}{n\pi}\left(-\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi}\right)\right].$$



$\Rightarrow \sin(n\pi) = 0$ for all n.

From the previous question, $\cos(n\pi) = (-1)^n$. So,

$$a_n = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right], \text{ where } a_0 = 0.$$

Hence, our Fourier series is:

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{(n\pi)^2}\cos(n\pi x).$$

Ex. Find the Fourier series for

$$f(x) = x^2, \quad -1 < x < 1.$$

Notice that f is even, and so

$$f(x)\sin(n\pi x) \text{ is odd,}$$

and so $b_n = 0$.

Then, since $f(x)\cos(n\pi x)$ is even,

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx$$
$$= 2\int_0^1 x^2\cos(n\pi x)dx, \qquad u = x^2,\ du = 2x,\quad dv = \cos(n\pi x),\ v = \frac{\sin(n\pi x)}{n\pi}$$
$$= 2\left[\frac{x^2\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)\right]$$

$\sin(n\pi) = 0$

$$= -\frac{4}{n\pi}\int_0^1 x\sin(n\pi x)dx, \qquad u = x,\ du = 1,\quad dv = \sin(n\pi x),\ v = -\frac{\cos(n\pi x)}{n\pi}$$
$$= -\frac{4}{n\pi}\left[-\frac{x\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1\cos(n\pi x)dx\right]$$
$$= -\frac{4}{n\pi}\left[-\frac{\cos(n\pi)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\Big|_0^1\right]$$

$\sin(n\pi) = 0$

$$= \frac{4\cos(n\pi)}{(n\pi)^2}$$
$$= \frac{4(-1)^n}{(n\pi)^2}.$$

Finally,

$$a_0 = \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = 2\frac{x^3}{3}\Big|_0^1$$
$$= \frac{2}{3}.$$

So, our Fourier series is:

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{(n\pi)^2}\cos(n\pi x)$$