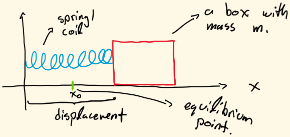
Bonus Lab

Tuesday, October 1, 2019 3:20 PM

Topics:
- Free mechanical vibrations.
- Damped vibrations.

General Vibrations:

When it comes to physical applications, I always find pictures useful. When looking at free vibration or oscillation, I imagine the following setup:



If our box is beyond this equilibrium point x_0, we expect the spring to pull the box towards the wall.

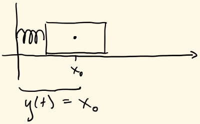
If our box is located in front of the equilibrium point, we expect the spring to push the box away from the wall.

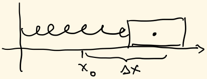
In this way, if we displace our box from its natural resting point, it should "wiggle" back and forth around this point. But how does it wiggle? Does it stop? Does it go forever? Let's find out!

As is tradition in physics, we start with good ol' Newton, who asserts that

$$F = ma.$$

That is, force is equal to mass & acceleration. Then, let's take our function $y(t)$ to be the function which describes the displacement of the box from the equilibrium point x_0 at time t. See the new picture:





$$\underbrace{\qquad\qquad}_{y(t) = x_0 + \Delta x}$$

Then, recall that the second derivative of displacement is equal to acceleration:

$$\frac{d^2}{dt^2}(\text{displacement}) = \frac{d^2y}{dt^2}$$

$$= a .$$

If we combine this with Newton's law:

$$F = ma \Rightarrow m\frac{d^2y}{dt^2} = F .$$

So, in order to solve for y, we need to figure out which external forces act on our spring-box setup. First, as described earlier, there is the force of the spring. Intuitively, it makes sense that the strength of the force is proportional to the amount of displacement:

Lots of displacement => lots of force
Little displacement => little force.

This is exactly Hooke's Law, which says

$$F_{\text{spring}} = -Ky(t),$$

where K is a constant depending on the material properties of the spring. Here, we have a negative sign which denotes the opposing nature of the force.

Then, there is also the force of friction between the box and the surface it is on. It is known (not by me, but the physicists say so and I believe them) that the force of friction is proportional to velocity:

$$F_{\text{fric.}} = -b\,(\text{velocity})$$

$$= -b \cdot \frac{dy}{dt} ,$$

where we have used that the first derivative of displacement is velocity. Again, the coefficient b relates to the level of friction between the box and the surface, and is negative to denote the opposing nature of the force.

If we combine these forces, we arrive at a differential equation:

$$ma = F \Rightarrow \quad m\frac{d^2y}{dt^2} = F_{\text{spring}} + F_{\text{fric.}}$$

$$= -ky - b\frac{dy}{dt}.$$

Thus,

$$\boxed{my'' + by' + ky = 0}$$

This is the general equation for a vibrating object with no external forces.

Example: Suppose we have an object with mass 1 kg, and has no friction. Then, our equation becomes

$$1 \cdot y'' + ky = 0.$$

Remember that the spring constant k is always positive. Then, our characteristic equation is

$$r^2 + k = 0$$

$$\Rightarrow \quad r = \pm i\sqrt{k}.$$

Then, our solution is $y(t) = C_1\cos(\sqrt{k}\,t) + C_2\sin(\sqrt{k}\,t)$.

This makes sense, as it suggests that under no friction, the box will continue to oscillate forever without slowing down. This is an undamped equation.

Example: Suppose now that we are given a spring with constant $k = 1$ for the same box, and now include friction with coefficient $b = 1$. Then, we must solve

$$y'' + y' + y = 0, \quad \text{and so}$$

$$r^2 + r + 1 = 0 \implies \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2}$$

$$\implies r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Then,

$$y(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\tfrac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\tfrac{\sqrt{3}}{2}t\right)\right).$$

This is called a damped equation, since

$$\lim_{t \to \infty} y(t) = 0.$$

That is, friction is slowing down, or "damping" our solution as time goes on.

Free Vibrations:

Now we can discuss a simple case: $b = 0$. In this case, there is zero friction, and so the vibrations are free to continue indefinitely. In this case, we have

$$my'' + Ky = 0, \quad \text{or}$$

$$y'' + \left(\frac{K}{m}\right)y = 0.$$

If we set $\omega^2 = \frac{K}{m}$, this

becomes

$$y'' + \omega^2 y = 0$$

with roots $r = \pm i\omega$. Our solutions are then

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

where $\omega = \sqrt{\frac{K}{m}}$.

Using some trig identities, we can write this as a single trig function:

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi),$$

where

$$A = \sqrt{c_1^2 + c_2^2}, \quad \text{and}$$

$$\phi = \arctan\left(\frac{C_2}{C_1}\right).$$

This is achieved by taking

$$C_1 = A \sin\phi,$$
$$C_2 = A \cos\phi,$$

and using the addition formula for sin:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

So, for free mechanical vibrations, the solution is given by

$$y(t) = A \sin(\omega t + \phi).$$

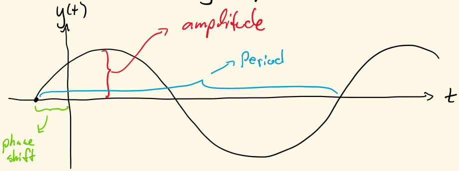
This is convinient, since

$$A = \text{amplitude},$$

$$\frac{2\pi}{\omega} = \text{Period} = \frac{1}{\text{Frequency}},$$

$$\phi = \text{phase shift}.$$

See the following picture:



Amplitude is just the amount of displacement from the equilibrium.

Period is the amount of time it takes to Complet one cycle or oscillation.

Phase shift is the amount we are shifted to start.

Overdamped, Underdamped, Critically damped

If we think back to our general, homogeneous, second order, constant coefficient equation, we had 3 important cases:

$$\left.\begin{aligned} b^2-4mk &> 0, \\ b^2-4mk &= 0, \\ b^2-4mk &< 0. \end{aligned}\right\} my''+by'+ky=0$$

If our vibrating system is no longer free (i.e. there is some friction), the nature of the motion described depends on the three cases above.

Case I: Overdamped corresponds to the case when $b^2-4ac>0$. In this case, both roots to our auxilary equation are real and distinct. Furthermore, they are both negative, since

$$\frac{-b-\sqrt{b^2-4mk}}{2m} < 0, \qquad \text{and}$$

$$\sqrt{b^2-4mk} < b \Rightarrow \frac{-b+\sqrt{b^2-4mk}}{2m} < 0.$$

Thus, our solution looks like

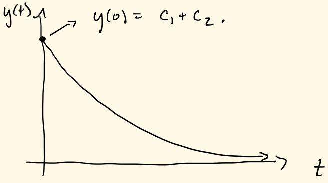
$$y(t) = c_1e^{r_1t} + c_2e^{r_2t},$$

and when $r_1, r_2 < 0$,

$$\lim_{t\to\infty} y(t) = 0.$$

In this case, no oscillation occurs, and the system is called overdamped.

Picture:



Case II: Underdamped corresponds to the case when $b^2-4mk<0$. Solutions are then given by

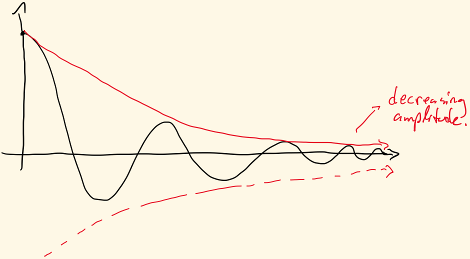
$$y(t) = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right),$$

where $\alpha = \frac{-b}{2m} \quad (< 0)$

$$\beta = \frac{\sqrt{|b^2-4mk|}}{2m}.$$

In this case, the solutions oscillate, but the oscillations decrease in amplitude as time goes on. This is called underdamped.

Picture:



Case III: Critically damped is the special case when $b^2-4mk=0$. In this case we have one negative, repeated root:

$$r = \frac{-b}{2m}.$$

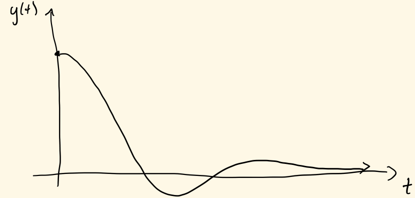
Solutions are given by $y(t) = c_1e^{rt} + c_2te^{rt}$.

One can show that we still have

$$\lim_{t\to\infty} y(t) = 0,$$

and the solution may cross the t-axis, but the motion is non oscillatory, since it only has at most two turning points:

Picture:



Practice Problem: Consider the mass spring equation

$$\begin{cases} y'' + by' + 25y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solve the equation for $b = 6, 10$ and 12 and classify each solution as one of the three cases discussed above. Plot each solution.