Lab Eight

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Topics:
- Laplace with convolutions
- Laplace with delta 'functions'
- Systems of ODE's

Convolutions: The end goal here is to use properties of convolutions, combined with the Laplace Transform, to solve some stuff.

To do so, we need to know what convolution means in this context.

Given two functions $f(t)$ and $g(t)$ which are piecewise continuous on $[0,\infty)$, the convolution of f and g, denoted $f*g$, is defined to be:

$$\boxed{(f*g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau\ .}$$

Ex. Find the convolution of

$$f(t) := t\ , \quad g(t) = t^2.$$

By definition,

$$\begin{aligned}(f*g)(t) &:= \int_0^t f(t-\tau)\,g(\tau)\,d\tau \\ &= \int_0^t (t-\tau)\cdot\tau^2 \\ &= t\int_0^t \tau^2 d\tau - \int_0^{\infty} \tau^3 d\tau \\ &= t\ \frac{\tau^3}{3}\bigg|_0^t - \frac{\tau^4}{4}\bigg|_0^t \\ &= \frac{t^4}{3} - \frac{t^4}{4} \\ &= \frac{t^4}{12}\ .\end{aligned}$$

Now, $*$ does not denote normal multiplication as we know it. However, the convolution has some nice properties.

Properties of Convolutions:

$$\boxed{\begin{array}{ll} 1. & f*g = g*f \\ 2. & f*(g+h) = f*g + f*h \\ 3. & f*(g*h) = (f*g)*h \\ 4. & f*0 = 0 \end{array}}$$

Why do we even care??

What follows is the so-called convolution theorem, which relates convolutions and the Laplace transform.

Convolution Theorem: Suppose that

$$\mathcal{L}\{f(t)\}(s) = F(s) \quad \text{and} \quad \mathcal{L}\{g(t)\}(s) = G(s).$$

Then,

$$\boxed{\mathcal{L}\{f * g\}(s) = F(s)\,G(s),}$$

or equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t)\cdot g(t).$$

What this tells us is that the convolution of functions in the variable t results in the multiplication of functions in the variable s.

Ex. Solve the following IVP:

$$\begin{cases} y'' - y = g(t), \\ y(0) = 1, \\ y'(0) = 1, \end{cases}$$

assuming that $\mathcal{L}\{g(t)\}(s) = G(s)$ exists.

As usual, take the LT of both sides:

$$\mathcal{L}\{y'' - y\} = s^2 Y(s) - s - 1 - Y(s)$$

$$= \mathcal{L}\{g(t)\} = G(s).$$

Then, solve for $Y(s)$:

$$Y(s) = \frac{s+1 + G(s)}{s^2 - 1}$$

$$= \frac{1}{s-1} + \frac{G(s)}{s^2-1}.$$

Then, we invert to solve for $y(t)$:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{G(s)}{s^2-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2-1}\right\} \\ &= e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\cdot G(s)\right\}. \end{aligned}$$

Now, to solve for the last piece, we will use the convolution theorem. According to the theorem,

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f * g,$$

and so if $F(s) = \frac{1}{s^2-1}$,

we only need to find

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}.$$

Looking at our table of L-T's, we see that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh(t).$$

Then, $\mathcal{L}^{-1}\{F(s)G(s)\}$

$$\begin{aligned} &= f * g \\ &= \sinh(t) * g(t) \\ &= \int_0^t \sinh(t-\tau)\, g(\tau)\, d\tau . \end{aligned}$$

Thus, our final solution is:

$$y(t) = e^t + \int_0^t \sinh(t-\tau)\, g(\tau)\, d\tau ,$$

for any $g(t)$ that is "nice". → Here, when I say "nice", I mean that the Laplace Transform of g exists.

Delta Functions: Now we define this weird object we call a delta "function" (which is not a function at all), but can be defined by the following two properties:

The Delta Function, $\delta(t)$, is a "function" such that

1. $$\delta(t) := \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0),$$

   for any continuous function $f(t)$ on an interval containing zero.

So, a delta function is zero everywhere except at $t=0$, and integration of $\delta(t)$ multiplied by a "nice" function $f(t)$ extracts the function value at $t=0$.

Relation to the Laplace Transform:

Often in ODE's, you have an event that happens at an instant. In mathematical terms, if the event happens at $t=a$, then we use

$$\delta(t-a), \quad a > 0$$

to denote this instantaneous event, occuring at $t=a$.

Then, we need to know $\mathcal{L}\{\delta(t-a)\}$. This is easy. By definition of the L-T,

$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^{\infty} e^{-st}\delta(t-a)\, dt,$$

and since $\delta(t-a) = 0$ for all $t < a$,

$$= \int_{-\infty}^{\infty} e^{-st}\,\delta(t-a)\, dt,$$

and by the definition of this delta function,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a),$$

and so

$$\mathcal{L}\{\delta(t-a)\} = \int_{-\infty}^{\infty} e^{-st} f(t-a)\,dt$$
$$= e^{-as}.$$

Ex. Solve the IVP:

$$\begin{cases} x'' + 9x = 3\delta(t-\pi) \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$$

First, take the L-T of both sides:

$$\mathcal{L}\{x''+9x\} = s^2X(s) - s + 9X(s)$$
$$\mathcal{L}\{3\delta(t-\pi)\} = 3e^{-\pi s}$$

Then solve for $X(s)$:

$$X(s) = \frac{s}{s^2+3^2} + e^{-\pi s}\frac{3}{s^2+3^2}$$

Finish by inverting $X(s)$ to solve for $x(t)$.

We already know that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos(3t).$$

Furthermore, $\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin(3t)$.

Thus, since $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$,

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{3}{s^2+3^2}\right\} = \sin(3(t-\pi))\,u(t-\pi)$$

Thus, our final solution is:

$$x(t) = \cos(3t) + \sin(3(t-\pi))\,u(t-\pi).$$

Systems of ODE's

It turns out that the Laplace Transform is also useful when applied to systems.

The general strategy is the same and one solves for each solution separately.

Ex. Solve the system:

$$\begin{cases} x' - 2y = 4t \;; & x(0) = 4 \\ y' + 2y - 4x = -4t - 2 \;; & y(0) = -5 \end{cases}$$

1. Take the LT of both equations:

$$\mathcal{L}\{x' - 2y\} = sX(s) - x(0) - 2Y(s)$$
$$= sX - 4 - 2Y,$$

$$\mathcal{L}\{y' + 2y - 4x\} = sY + 5 + 2Y - 4X,$$

$$\mathcal{L}\{4t\} = 4\cdot\frac{1!}{s^{1+1}} = \frac{4}{s^2},$$

$$\mathcal{L}\{-4t-2\} = -\frac{4}{s^2} - \frac{2}{s}$$

So, we now solve for X and Y from:

$$\begin{cases} sX - 2Y = 4 + \frac{4}{s^2} & \textcircled{I} \\ -4X + (2+s)Y = -\frac{4}{s^2} - \frac{2}{s} - 5 & \textcircled{II} \end{cases}$$

Now we solve for X and Y. Consider:

$$(s+2)\textcircled{I} + 2\textcircled{II}$$

$$= \left[s(s+2)X - \cancel{2(s+2)Y}\right] + \left[-8X + \cancel{2(s+2)Y}\right] \boxed{=} \frac{4(s+2)(s^2+1)}{s^2} + \frac{2}{s^2}\left(-5s^2 - 2s - 4\right)$$

$$\hookrightarrow = X\left[s^2 + 2s - 8\right] = \frac{1}{s^2}\left(4(s+2)(s^2+1) - 2(5s^2+2s+4)\right)$$

$$\hookrightarrow X\left[(s+4)(s-2)\right] = \frac{1}{s^2}\left(4(s+2)(s^2+1) - 2(5s^2+2s+4)\right)$$

$$\hookrightarrow X(s) = \frac{4s-2}{(s+4)(s-2)}$$

PARTIAL FRAC'S:

$$X(s) = \frac{3}{s+4} + \frac{1}{s-2}.$$

We can now invert X to obtain x(t):

$$x(t) = \mathcal{L}^{-1}\left\{\frac{3}{s+4} + \frac{1}{s-2}\right\}$$

$$= 3e^{-4t} + 1 \cdot e^{2t}$$

Now we could repeat the same for Y(s), or we could solve for y(t) using equation ①:

$$x' - 2y = 4t, \quad \text{and so}$$

$$y(t) = \frac{x'}{2} - 2t$$

$$= \frac{-12e^{-4t}}{2} + 2e^{2t} - 2t$$

$$= -6e^{-4t} + 2e^{2t} - 2t.$$

And we're done!