Lab Five

Tuesday, October 8, 2019 12:31 PM

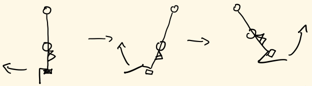
Topics:
- Forced Vibration (resonance)
- Cauchy - Euler equations
- Reduction of order

Forced Vibrations

After reading the bonus lab on free vibrations, "forced" vibration is a very similar equation, but this time the equation is inhomogeneous. That is,

$$my'' + by' + Ky = G(t), \qquad ⊛$$

where $G(t)$ is some external forcing. As an example, one could consider someone on a swing set:



They are swinging (or vibrating) back and forth, and the external force would be the "pumping" given by your legs.

Example: Consider a periodic forcing term

$$G(t) := A\cos(\gamma t),$$

where A and γ are constants.

From work done in the previous lab, we know that the homogeneous solution is given by:

$$y_h(t) = Ce^{-\left(\frac{b}{2m}\right)t}\sin\left(\frac{\sqrt{4mK-b^2}}{2m}t + \phi\right),$$

where C is constant and ϕ is our phase shift.

We can then solve for the particular solution using undetermined coefficients. That is, guess

$$y_p(t) = A_1\cos(\gamma t) + A_2\sin(\gamma t).$$

Plugging this into ⊛ and solving for A_1 and A_2 (I encourage you to do this yourself!) gives us:

$$y_p(t) = \frac{A}{(K-m\gamma^2)^2 + b^2\gamma^2}\left((K-m\gamma^2)\cos(\gamma t) + b\gamma\sin(\gamma t)\right)$$

Using some trig identities, this can also be written as

$$y_p(t) = \frac{A}{\sqrt{(K-m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \theta),$$

where θ is a new phase shift (of the particular solution) given by

$$\theta = \arctan\left(\frac{K-m\gamma^2}{b\gamma}\right).$$

Our general solution is then given by the sum of y_p and y_h, which is valid for $0 < b < 4mK$. (Otherwise we don't have oscillations!)

Resonance

This particular solution found is what remains as $t \to \infty$, since $y_h(t) \to 0$ as $t \to \infty$, and our particular solution has no decay as $t \to \infty$. Then, the amplitude of the particular solution is determined by

$$M(\gamma) := \frac{1}{\sqrt{(K-m\gamma^2)^2 + b^2\gamma^2}}.$$

which depends heavily on the frequency of our forcing term, which is γ.

This leads to the idea of resonance.

If the frequency of the forcing is is close to the natural frequency of the problem, and the damping coefficient of b is small, the amplitude of the particular solution can become very large.

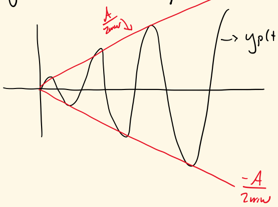
In particular, if the natural frequency $\omega = \sqrt{\frac{k}{m}}$ is the same as the forcing frequency γ, our particular sol$^{\underline{n}}$ is then a part of the homogeneous sol$^{\underline{n}}$!! This means that in order to find the new particular sol$^{\underline{n}}$, we have to guess

$$y_p = A_1 t\cos(\omega t) + A_2 t\sin(\omega t),$$

which gives us

$$y_p(t) = \frac{A}{2m\omega} t\sin(\omega t),$$

which grows linearly as $t \to \infty$!



Cauchy-Euler Equations

Consider the following ODE:

$$x^2y'' + bxy' + cy = 0,$$ (this is in standard form!)

where a, b, c are constants. This equation is second order and linear, but we cannot solve it using methods learned so far. This is called a Cauchy-Euler equation.

Solving C-E Eq$^{\underline{n}}$'s

As usual, we make a "guess" and see what happens. Because of the powers of x appearing as coefficients, we try:

$$y(x) = x^m,$$

for some m to be determined. Then,

$$y'(x) = m x^{m-1},$$
$$y''(x) = m(m-1)x^{m-2}.$$

So, if we plug this into our equation, we have:

$$\begin{aligned} 0 &= x^2y'' + bxy' + cy \\ &= x^2[m(m-1)x^{m-2}] \\ &\quad + bx[mx^{m-1}] \\ &\quad + cx^m \\ &= x^m[\ m(m-1) + bm + c] \\ &= x^m[\ m^2 + (b-1)m + c] \end{aligned}$$

In order for this to be zero for all x,

we require that

$$m^2 + (b-1)m + c = 0,$$

which is just a quadratic function of m!

So,
$$m = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4c}}{2} .$$

As in the constant coefficients case, we have 3 cases:

Case I: Two distinct roots m_1 and m_2.

Case II: One real repeated root m_1.

Case III: Complex conjugates $m = \alpha \pm i\beta$,

where $\alpha = -\frac{(b-1)}{2}$,

$$\beta = \frac{\sqrt{4c-(b-1)^2}}{2} .$$

Case I: In this case, our fundamental solution is simply:

$$\boxed{y_h(x) = c_1 x^{m_1} + c_2 x^{m_2}}$$

Case II: In this case, we have one root, and so $c_1 x^m$ is the first piece of the homogeneous solution. As in the constant coefficient case, we must multiply our first piece by some function, but this time, choosing a factor of x does not work! Instead, it turns out that the proper function is $\ln(x)$, and so our homogeneous solution is

$$\boxed{y_h(x) = c_1 x^m + c_2 \ln(x) x^m .}$$

Case III: In this final case, we have complex roots $\alpha \pm i\beta$. Similar to the constant coefficients case, we have a factor involving α, and sine & cosine terms involving β. The homogeneous solution is:

$$\boxed{y_h(x) = x^{\alpha}\left(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))\right)}$$

To Show This: Case I is easy and involves no extra work. For cases II and III, use the change of variables

$$x = e^t, \quad \text{or}$$
$$\ln(x) = t, \quad \text{and look for a sol}^{\underline{n}}$$
$$y(x) = \phi(t).$$

Plugging this in gives $\phi'' + (b-1)\phi' + c\phi = 0$,

and sol$^{\underline{n}}$'s are $\phi(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$,

where λ_1, λ_2 are the roots of

$$r^2 + (b-1)r + c = 0,$$

then undo the original substitution. If you want more details for the derivation, email me!!

Ex1. Solve $x^2 y'' + 3xy' + 4y = 0$

Roots are given by $r^2 + (b-1)r + c = 0$, where $b = 3$, $c = 4$. So,

$$r^2 + 2r + 4 = 0, \qquad \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 4}}{2}$$

$$= -1 \pm \frac{i\sqrt{12}}{2}$$

$$= -1 \pm i\sqrt{3}.$$

So, we are in case III, with $\alpha = -1$, $\beta = \sqrt{3}$.

Thus,

$$y_h(x) = x^{-1}\left(C_1 \cos(\sqrt{3}\ln(x)) + C_2 \sin(\sqrt{3}\ln(x))\right).$$

Ex 2. Solve

$$x^2y'' + 3xy' + y = 0.$$

Roots are: $r^2 + (b-1)r + c = 0$

So, $r^2 + 2r + 1 = 0$

$$\Rightarrow \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 1}}{2}$$

$\Rightarrow r = -1$, repeated.

We are in case II, and so

$$y_h(x) = C_1 x^{-1} + C_2 \ln(x) x^{-1}.$$

Reduction of Order

Sometimes, when you are given a higher order equation, you can reduce the order by making a special substitution. Although the order is reduced, the result is (usually) a system of ODE's which must then be solved. Sometimes this makes our lives easier, sometimes it does not.

Example: Suppose you can write your ODE as:

$$y'' = F(x, y'),$$

with no dependence on y. Then, substitute

$$\boxed{w = y'.}$$

Then,

$$y'' = F(x, w).$$

But, if we take a derivative of w,

$$w' = y'',$$ and so

$$\boxed{w' = F(x, w)}$$

Hence, we have reduced the second order equation into two first order equations:

$$\begin{cases} w' = F(x,w) \\ y' = w \end{cases}$$

So, if we can find w from the first equation, we can then find y from the second equation.

Ex. Solve

$$2xy'' - y' + \frac{1}{y'} = 0, \quad x > 0.$$

First, we write it in the standard form given above:

$$y'' = \frac{1}{2x}\left(y' - \frac{1}{y'}\right)$$

Then, we substitute

$$w = y',$$
$$w' = y'',$$ so that

$$w' = \frac{1}{2x}\left(w - \frac{1}{w}\right).$$

This equation is separable, and so

$$\frac{w\,dw}{w^2 - 1} = \frac{dx}{2x},$$

and if we integrate:

$$\int \frac{w}{w^2-1}\,dw = \frac{1}{2}\int \frac{1}{x}\,dx$$
$$= \frac{1}{2}\ln(x) + C$$
$$= \ln(\sqrt{x}) + C .$$

$$\downarrow$$

sub $w^2 - 1 = v$,
$\frac{dv}{2} = w\,dw$,

$$= \frac{1}{2}\int \frac{1}{v}\,dv = \frac{1}{2}\ln(v)$$
$$= \ln(\sqrt{w^2-1}) .$$

Hence, $\ln(\sqrt{w^2-1}) = \ln(\sqrt{x}) + C$,

or $\sqrt{w^2-1} = e^{C}\sqrt{x}$, or

$w^2 - 1 = e^{2C}x$, or

$w = \pm\sqrt{(C_0 x + 1)}$, where $C_0 = e^{2C}$.

Now, we know that $y'(x) = w(x)$, and so

$$\frac{dy}{dx} = \pm\sqrt{(C_0 x + 1)} ,$$ and if we integrate:

$$y(x) = \pm\int\sqrt{(C_0 x + 1)} ,$$
$$= \pm\frac{3}{2C_0}(C_0 x + 1)^{3/2} + D$$

Example 2: Suppose we have an ODE of the form

$$y'' = F(y, y').$$

Then, if we do the same trick,

$$y' = w ,$$
$$y'' = w' .$$

But, we now do an extra step. This time, write $\frac{d^2y}{dx} = \frac{dw}{dx} = \frac{dw}{dy}\cdot\frac{dy}{dx}$

$$= w\cdot\frac{dw}{dy} .$$

Then, our equations to solve are:

$$\begin{cases} w\,\dfrac{dw}{dy} = F(y, w) \\ \dfrac{dy}{dx} = w \end{cases}$$

In the first equation, we seek a solution $w := w(y)$, and so y is now the independent variable. Then, we find $y(x)$ through

$$\frac{dy}{dx} = w(y(x)) .$$

Ex. Solve $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

Standard form: $\frac{d^2y}{dx^2} = -y\frac{dy}{dx}$,

and so we must solve

$$\cancel{w}\,\frac{dw}{dy} = -y\cdot\cancel{w}$$

$$\Rightarrow \frac{dw}{dy} = -y$$

$$\Rightarrow w(y) = -\frac{y^2}{2} + C$$

Then, we solve for y:

$$\frac{dy}{dx} = w$$
$$= C - \frac{y^2}{2} .$$

This is separable, so

$$\int \frac{dy}{C - \frac{y^2}{2}} = \int dx$$
$$= x + D$$
$$\downarrow$$
$$= \sqrt{2}\operatorname{arctanh}\left(\frac{y}{\sqrt{2C}}\right) ,$$

and so

$$\operatorname{arctanh}\left(\frac{y}{\sqrt{2C}}\right) = \frac{x + D}{\sqrt{2}}$$

and so

$$y(x) = \sqrt{2C}\tanh\left(\frac{x + D}{\sqrt{2}}\right) .$$
