Lab Four

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Topics:
- Variation of Parameters
- Examples

Variation of Parameters

Suppose we have the following second order, linear ODE:

$$y'' + P(x)y' + Q(x)y = R(x), \qquad \circledast$$

for some functions P, Q, R. Similar to the method of undetermined coefficients, if we know the two linearly independent homogeneous solutions, we can determine the particular solution using the "variation of parameters". So how do we achieve this?

A Brief Overview: Suppose we have two homogeneous solutions $y_1(x)$ and $y_2(x)$. We try to find two new functions $u_1(x)$ and $u_2(x)$ so that the particular solution is of the form

$$y_p(x) = u_1 y_1 + u_2 y_2 .$$

We now proceed as usual: take some derivatives, make some simplifications and find a way to solve for u_1 and u_2. So, the first derivative is:

$$\begin{aligned} y_p'(x) &= u_1'y_1 + u_1 y_1' + u_2'y_2 + u_2 y_2' \\ &= (u_1'y_1 + u_2'y_2) + (u_1 y_1' + u_2 y_2'). \end{aligned}$$

Since we (in theory) know more about derivatives of y_1 and y_2, we simplify the above by assuming

$$\boxed{u_1'y_1 + u_2'y_2 = 0}.$$

Then, our second derivative of $y_p(x)$ is:

$$
\begin{aligned}
y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \\
&= (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2'')
\end{aligned}
$$

Then, if we plug in y_p, y_p' and y_p'' into our original equation $(*)$, we have that

$$\boxed{u_1'y_1' + u_2'y_2' = R(x)}$$

Note: I haven't written out these computations to keep things brief. I encourage you to try for yourself to see the magic happen.

Then, we have a system of equations to solve:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = R(x) \end{cases}$$

One can think of this as "two equations, two unknowns" (remember that u_1 and u_2 are unknown, y_1 and y_2 are already known), and so we should be able to solve fully for u_1 and u_2. Indeed, one can substitute

$$u_1' = -\frac{u_2'y_2}{y_1}$$

from the first equation into the second equation to find

$$u_2' = -\frac{y_1 R(x)}{y_1y_2' - y_1'y_2},$$

and putting this back into the first equation yields

$$u_1' = \frac{y_2 R(x)}{y_1 y_2' - y_1' y_2}$$

We can then integrate to obtain u_1 and u_2:

$$u_1(x) = -\int \frac{y_2 R(x)}{y_1 y_2' - y_1' y_2}\, dx$$

$$u_2(x) = \int \frac{y_1 R(x)}{y_1 y_2' - y_1' y_2}\, dx \,.$$

As a final note, recall the Wronskian:

$$\begin{aligned} W(y_1, y_2) &= \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &\neq 0 \,, \quad \forall x , \end{aligned}$$

since $y_1(x)$ and $y_2(x)$ are fundamental solutions of problem $\circledast$. Therefore the integrals above make sense (we haven't divided by zero), and can then be written as:

$$\boxed{\begin{aligned} u_1(x) &= -\int \frac{y_2 R(x)}{W(y_1, y_2)}\, dx \\ u_2(x) &= \int \frac{y_1 R(x)}{W(y_1, y_2)}\, dx \end{aligned}}$$

Examples:

Below we can find some examples which use these formulas found above.

Ex. 1. Solve

$$y'' + 2y' + y = e^{-x}\sin(x).$$

Step 1. We need the homogeneous solution, so use the characteristic eq$^{\underline{n}}$!

$$r^2 + 2r + 1 = 0$$

$$\Rightarrow \quad r = -1, \text{ repeated. So,}$$

$$\begin{cases} y_1(x) = c_1 e^{-x}, \\ y_2(x) = c_2 x e^{-x}. \end{cases}$$

Step 2. We know we have plug information into the integrals above, so I first compute $W(y_1, y_2)$:

$$W(y_1,y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$

Notice I don't bother with the constants. Why is that?

$$= e^{-x}(e^{-x} - xe^{-x}) + e^{-x}(xe^{-x})$$

$$= e^{-2x}.$$

Now we solve for u_1 and u_2:

$$u_1 = -\int \frac{y_2 R(x)}{W(y_1,y_2)}\,dx$$

$$= -\int \frac{x\cancel{e^{-x}} \cdot \cancel{e^{-x}} \sin x}{\cancel{e^{-2x}}}\,dx$$

$$= -\int x \sin x \, dx, \qquad \begin{matrix} u = x & dv = \sin x \\ du = dx & v = -\cos x \end{matrix}$$

$$= -\left(-x\cos x - \int -\cos x \, dx\right)$$

$$= x\cos x - \sin x$$

Then for u_2:

$$u_2(x) = +\int \frac{y_1 R(x)}{W(y_1,y_2)}$$

$$= \int \frac{e^{-x} e^{-x} \sin x}{e^{-2x}}$$

$$= \int \sin x$$

$$= -\cos x.$$

Step 3. Put this all together. Remember that our particular solution looks like

$$y_p(x) = y_1 u_1 + y_2 u_2,$$

and so our final solution is:

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 e^{-x} + c_2 x e^{-x} + e^{-x}(x\cos x - \sin x) + x e^{-x}(-\cos x)$$

$$= c_1 e^{-x} + c_2 x e^{-x} - e^{-x} \sin x$$

$//$

Ex. 2 Solve

$$y'' - 2y' + y = \frac{e^x}{x^2+1}.$$

Step 1. Hom. Solⁿ:

$$r^2 - 2r + 1 = 0$$

$$\Rightarrow \quad r = 1, \text{ repeated.}$$

Therefore

$$y_1 = c_1 e^x, \quad y_2 = c_2 x e^x.$$

Step 2. Wronskian:

$$W(y_1,y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^x(e^x + xe^x) - xe^x e^x$$

$$= e^{2x}.$$

Now find u_1 and u_2:

$$u_1(x) = -\int \frac{y_2 R(x)}{W(y_1, y_2)}$$

$$= -\int \frac{xe^x \cdot e^x}{e^{2x}(x^2+1)}$$

$$= -\int \frac{x}{x^2+1}, \quad \text{set} \quad u = x^2+1, \quad \frac{du}{2} = x\,dx$$

$$= -\frac{1}{2}\ln(x^2+1), \quad \text{and}$$

$$u_2(x) = +\int \frac{y_1 R(x)}{W(y_1, y_2)}\,dx$$

$$= \int \frac{e^x\ e^x}{e^{2x}\cdot(x^2+1)}\,dx$$

$$= \int \frac{1}{x^2+1}, \quad \text{this is } \arctan(x)\,! \ (\text{Use trig subs}).$$

$$= \arctan(x).$$

Step 3. Put it all together:

$$y(x) = y_h + y_p$$

$$= c_1 e^x + c_2 xe^x + e^x\left(-\frac{1}{2}\ln(x^2+1)\right) + xe^x(\arctan(x))$$

$$= c_1 e^x + c_2 xe^x - e^x \ln(\sqrt{x^2+1}) + xe^x \arctan(x).$$

Ex. 3. Solve

$$t y'' - (t+1) y' + y = t^2,$$

given that $y_1(t) = e^t$,

$y_2(t) = t+1$

are homogeneous solutions to the problem.

Step 1. This time we are given the hom. sol^{n}'s, but the equation is not in standard form.

So, assume $t \neq 0$ and rewrite as:

$$y'' - \left(\frac{t+1}{t}\right) y' + \left(\frac{1}{t}\right) y = t.$$

Step 2. Solve for u_1, u_2 with $R(t) = t$.

First, the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix}$$

$$= e^t - (t+1)e^t$$

$$= -te^t.$$

Then,

$$u_1(t) = -\int \frac{y_2 R(t)}{W(y_1,y_2)}\, dt$$

$$= + \int \frac{(t+1)\cdot \cancel{t}}{\cancel{-}\cancel{t}e^t}$$

$$= + \int (t+1) e^{-t} \quad , \qquad \begin{matrix} u = t+1 \\ du = 1 \end{matrix} \qquad \begin{matrix} dv = e^{-t} \\ v = -e^{-t} \end{matrix}$$

$$= \left(-(t+1)e^{-t} - \int (-e^{-t})\, dt \right)$$

$$= -(t+1)e^{-t} - e^{-t}$$

$$= -e^{-t}(t+2)$$

and

$$u_2(t) = \int \frac{y_1 R(t)}{W(y_1,y_2)}\, dt$$

$$= \int \frac{e^t\, t}{-te^t}$$

$$= -t.$$

Step 3. Our final solution is then:

$$y(t) = y_h + y_p$$

$$= c_1 e^t + c_2(t+1)$$

$$+ (-\cancel{e^{-t}}(t+2)) \cdot \cancel{e^t}$$

$$+ (-t)(t+1)$$

$$y(t) = c_1 e^t + c_2(t+1)$$

$$- t^2 - 2t - 2$$

$-t - 2$

$-t^2 - t$ $\quad -t^2 - 4t$

Ex 4. Solve

$$y'' + y = \tan x$$

(Can't be done with undetermined coefficients!)

Step 1. Hom. Sol^{n}.: $r^2 + 1 = 0$

$$\Rightarrow r = \pm i$$

So,
$$y_1(x) = c_1 \cos(x)$$
$$y_2(x) = c_2 \sin(x)$$

Step 2. Find u_1 and u_2. Start with $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1 \quad \text{(nice!)}$$

Then solve for u_1 and u_2:

$$u_1(x) = -\int \frac{y_2 R(x)}{W(y_1, y_2)}\, dx$$

$$= -\int \sin x \tan x\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx \quad , \qquad \sin^2 x = 1 - \cos^2 x$$

$$= -\int \frac{1-\cos^2 x}{\cos x}$$

$$= -\int \sec x \;+\; \int \cos x$$

$$= -\ln|\sec x + \tan x| \;+\; \sin x$$

Also,

$$u_2(x) = +\int \frac{y_1 R(x)}{W(y_1,y_2)}\,dx$$

$$= \int \cos(x)\tan x$$

$$= \int \sin x$$

$$= -\cos x.$$

Step 3. Our final solution is then

$$y(x) = y_h + y_p$$

$$= c_1\cos x + c_2 \sin x$$
$$+ (\cos x)\left(-\ln(\sec x + \tan x) \;\cancel{+\sin x}\right)$$
$$+ (\sin x)\left(\cancel{-\cos x}\right)$$

$$= c_1\cos x + c_2 \sin x$$
$$- \cos x \ln(\sec x + \tan x)$$