Lab Three

Monday, September 23, 2019 10:22 AM

Topics: - second order (homogeneous) equations
- undetermined coefficients (inhomogeneous)

1. Suppose we are given an ODE of the form

$$ay'' + by' + cy = 0$$

This is a second order ODE (highest derivative is second)

This is a linear ODE (all terms involving y or its derivatives appear linearly)

This is a homogeneous ODE (the right hand side is zero)

How can we solve this? Well, suppose we guess that the solution is of the form

$$y(x) = e^{rx},$$

for some value r to be determined.

Then,

$$y(x) = e^{rx},$$
$$y'(x) = re^{rx},$$
$$y''(x) = r^2e^{rx}.$$

If we plug this into our equation, we have:

$$\begin{aligned} 0 &= ay'' + by' + cy \\ &= ar^2e^{rx} + bre^{rx} + ce^{rx} \\ &= e^{rx}(ar^2 + br + c) \end{aligned}$$

Since this must hold true for all $x \in \mathbb{R}$, the only way this can be satisfied is if $ar^2 + br + c = 0$!

This is called the characteristic equation!

Using the quadratic formula, we know the solutions are:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let's call these two roots r_1 and r_2. The solution(s) to our original ODE are then

$$y_1(x) = e^{r_1x}, \quad y_2(x) = e^{r_2x}$$

Depending on the discriminant, we have three cases to consider:

Ⓘ $b^2 - 4ac > 0$ (two distinct, real roots r_1, r_2)

Ⓘⓘ $b^2 - 4ac = 0$ (one real, repeated root, $r_1 = r_2$).

Ⓘⓘⓘ $b^2 - 4ac < 0$ (two complex roots, which are conjugate)

Case I: Two Real, Distinct Roots:

In this case, we have two distinct roots which yield two (linearly independent) solutions:

$$y_1(x) = C_1 e^{r_1 x},$$
$$y_2(x) = C_2 e^{r_2 x},$$

where the two constants C_1, C_2 will be determined by some initial data. Our final solution is given by a linear combination of these two solutions:

$$\begin{aligned} y(x) &= y_1(x) + y_2(x) \\ &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \end{aligned}$$

Example: Solve $2y'' + 4y' - y = 0$.

First, find r_1, r_2, the roots of our characteristic equation: $2r^2 + 4r - 1 = 0$

$$\Rightarrow \frac{-4 \pm \sqrt{16 - 4(2)(-1)}}{2 \cdot 2}$$

$$\Rightarrow -1 \pm \frac{\sqrt{24}}{4}$$

$$\Rightarrow -1 \pm \frac{\sqrt{6}}{2}$$

Our final solution is then:

$$y(x) = c_1 e^{(-1 + \frac{\sqrt{6}}{2})x} + c_2 e^{(-1 - \frac{\sqrt{6}}{2})x}.$$

Case II: One Real Repeated Root

In this case, $b^2 - 4ac = 0$, and we only have one root... our solution is then

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_1 x} \\ &= (c_1 + c_2) e^{r_1 x} \\ &= C e^{r_1 x}, \end{aligned}$$

which is a problem, as this yields only one linearly independent solution! However, we can find a second solution simply by multiplying our existing solution by x:

$$y_1(x) = c_1 e^{r_1 x},$$
$$y_2(x) = c_2 x e^{r_1 x}.$$

Then, our final solution is:

$$y(x) = y_1(x) + y_2(x)$$
$$= c_1 e^{r_1 x} + c_2 x e^{r_2 x}.$$

How does one find this? See to the right:

Example: Solve $2y'' + 4y' + 2y = 0$

Characteristic Eq$^{\underline{n}}$: $\dfrac{-4 \pm \sqrt{16 - 4(2)(2)}}{2 \cdot 2}$

$$= -1 \pm 0$$
$$= -1.$$

We have one repeated root, and so

$$y_1(x) = c_1 e^{-x}, \quad \text{and}$$
$$y_2(x) = c_2 x e^{-x}.$$

Hence, our final solution is:

$$y(x) = y_1(x) + y_2(x)$$
$$= c_1 e^{-x} + c_2 x e^{-x}.$$

Case III: Two Complex Roots

In this final case, we have $b^2 - 4ac < 0$, and so $\sqrt{b^2 - 4ac}$ is complex valued.

Our roots are then:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b + i\sqrt{|b^2 - 4ac|}}{2a},$$
$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b - i\sqrt{|b^2 - 4ac|}}{2a}$$

Set

$$\alpha := \frac{-b}{2a},$$
$$\beta := \frac{\sqrt{|b^2 - 4ac|}}{2a}.$$

Our roots are then

$$r_1 = \alpha + i\beta,$$
$$r_2 = \alpha - i\beta.$$

Then, our solution(s) are given by:

$$y_1(x) = c_1 e^{(\alpha + i\beta)x}$$
$$y_2(x) = c_2 e^{(\alpha - i\beta)x}.$$

Here's where the magic happens!

Recall Euler's formula:

$$e^{it} = \cos(t) + i\sin(t).$$

We already have one solution: $y_1 = c_1 e^{rx}$, where $r = -\frac{b}{2a}$. Suppose we want a second (linearly independent) solution. We can compute the Wronskian, $W(y_1, y_2)$, in two ways:

Ⓘ $$W(y_1, y_2) = \det\begin{pmatrix} e^{rx} & y_2 \\ re^{rx} & y_2' \end{pmatrix}$$
$$= e^{rx} y_2' + re^{rx} y_2.$$

Ⓘⓘ $$W(y_1, y_2) = Ce^{-\int p(x)dx}$$
$$= Ce^{-\int \frac{b}{a}dx}$$
$$= Ce^{-\frac{b}{a}x}.$$

(This second form comes from the original eq$^{\underline{n}}$ in standard form:

$$y'' + \underset{\substack{\downarrow \\ p(t)}}{\left(\frac{b}{a}\right)} y' + \underset{\substack{\downarrow \\ q(t)}}{\left(\frac{c}{a}\right)} y = 0$$

Putting Ⓘ and Ⓘⓘ together yields:

$$e^{rx} y' + re^{rx} y = Ce^{-\frac{b}{a}x}$$

This is a first order, linear equation and can be solved using the integrating factor! If we solve this, the solution is

$$Cxe^{rx} + c_1 e^{rx}.$$

The second piece we already had, but the first piece is what we were looking for! Neat!

If $t = \beta x$, then

$$e^{it} = e^{i\beta x}$$
$$= \cos(\beta x) + i\sin(\beta x).$$

Then,
$$y_1(x) = c_1 e^{(\alpha+i\beta)x}$$
$$= c_1 e^{\alpha x} \cdot e^{i\beta x}$$
$$= c_1 e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$$

Similarly, $y_2(x) = c_2 e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$.

These solutions, while valid, are (in general) complex valued. If we set $c_1 = c_2 = \frac{1}{2}$, we can see that

$$V(x) = y_1(x) + y_2(x)$$
$$= e^{\alpha x}\cos(\beta x).$$

If we instead set $c_1 = \frac{1}{2i}$, $c_2 = -\frac{1}{2i}$, we have

$$u(x) = y_1(x) + y_2(x)$$
$$= e^{\alpha x}\sin(\beta x).$$

One can verify that these two solutions $V(x)$ and $u(x)$ are indeed linearly independent. Thus, our final solution is given by

$$y(x) = c_1 V(x) + c_2 u(x)$$
$$= c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$
$$= e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$$

Example: Solve $\frac{1}{2}y'' + y' + 5y = 0$

Characteristic Eqn: $\dfrac{-1 \pm \sqrt{1^2 - 4\cdot\frac{1}{2}\cdot 5}}{2\cdot\frac{1}{2}}$

$$= -1 \pm \sqrt{-9}$$
$$= -1 \pm i\cdot 3$$

So, $\alpha = -1$, $\beta = 3$.

Our solution is then

$$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$$
$$= e^{-x}(c_1\cos(3x) + c_2\sin(3x)).$$

Inhomogeneous Equations (Undetermined Coefficients)

Suppose now that we have an equation of the form

$$ay'' + by' + cy = Q(x),$$

for some function $Q(x) \neq 0$. The equation is still a linear, second order ODE, and so we still have the homogeneous sol$^{\underline{n}}$'s found in the previous section, depending on the characteristic eq$^{\underline{n}}$. But what about this $Q(x)$? In this case, we now need to find an additional solution, which we call the particular solution (in the sense that this new solution $y_p(x)$ is particular to the inhomogeneous piece $Q(x)$).

Depending on the form of $Q(x)$, we make a "guess" as to what the solution should look like and determine the coefficients of this guessed solution using the original equation.

A Simple Example: Solve

$$y'' + y' - 2y = x^2 .$$

Step 1: Find the homogeneous solution:

$$y_h'' + y_h' - 2y_h = 0$$

Characteristic Eq$^{\underline{n}}$:

$$\frac{-1 \pm \sqrt{1-4(1)(-2)}}{2}$$
$$= \frac{-1}{2} \pm \frac{\sqrt{9}}{2}$$
$$= \frac{-1}{2} \pm \frac{5}{2}$$

So we have two real, distinct roots:

$$\begin{cases} r_1 = 1 \\ r_2 = -2 \end{cases},$$

and so our homogeneous solution is:

$$y_h(x) = c_1 e^{x} + c_2 e^{-2x} .$$

Next, we find the particular solution. Notice that $Q(x) = x^2$. We will guess that $y_p(x) = Ax^2 + Bx + C$, a polynomial of order 2. Then,

$$y_p(x) = Ax^2 + Bx + C$$
$$y_p'(x) = 2Ax + B$$
$$y_p''(x) = 2A$$

If we plug this into our equation, we have

$$x^2 = y_p'' + y_p' - 2y_p$$
$$= 2A + (2Ax + B) - 2(Ax^2 + Bx + C)$$
$$= (-2A)x^2 + (2A - 2B)x + (2A + B - 2C)$$

We now solve for A, B, C by matching the coefficients of the left and right hand side:

$$\begin{cases} -2A = 1 \\ 2A - 2B = 0 \\ 2A + B - 2C = 0 \end{cases}$$

So,

$$A = \frac{-1}{2} \Rightarrow B = A = -\frac{1}{2},$$

and

$$\begin{aligned} C &= (2A + B)/2 \\ &= -1/2 - 1/4 \\ &= -3/4 \end{aligned}$$

Thus, $y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.

Our final solution is then given by

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{x} + c_2 e^{-2x} - \left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{3}{4}\right) \end{aligned}$$

This was an easy example, since the "guessed" particular solution was already linearly independent from the homogeneous solution. This is not always the case, unfortunately!

Example: What if instead we had:

$$y'' + y' - 2y = 4e^{x} \quad .$$

Similar to the previous example, our hom. sol$^{=}$ is given by $y_h(x) = c_1 e^{x} + c_2 e^{-2x}$.

If we try to "guess" the particular solution to be $y_p(x) = Ae^{x}$,

this is just our homogeneous sol$^{=}$, which will get us nowhere. (If you don't see why, try it!).

In such a case, the general strategy is to multiply by polynomials until the "guessed" solution is linearly independent from the homogeneous solution. So, in this case a polynomial of order 1 is sufficient. That is, guess:

$$y_p(x) = \underbrace{(Ax+B)}_{\substack{1^{st}\text{ order} \\ \text{poly}^{\underline{n}},\text{ since} \\ y_h(x)=c_1e^x}} \underbrace{e^x}_{\substack{\text{form of} \\ Q(x)}}$$

Now plug this into the equation and try to solve for A and B.

$$y_p(x) = (Ax+B)e^x$$
$$y_p'(x) = Axe^x + Ae^x + Be^x$$
$$y_p''(x) = Ae^x + Axe^x + Ae^x + Be^x$$

So, $y_p(x)$ must satisfy

$$\begin{aligned} 4e^x &= y_p'' + y_p' - 2y_p \\ &= (2Ae^x + Be^x + Axe^x) \\ &\quad + (Axe^x + Ae^x + Be^x) \\ &\quad -2(Axe^x + Be^x) \\ &= (2A + \cancel{B} + A + \cancel{B} - 2\cancel{B})e^x \\ &\quad + \underbrace{(A + A - 2A)}_{=0}xe^x \\ &= 3Ae^x. \end{aligned}$$

So, of course $B =$ anything "works", since Be^x is a homogeneous sol$^{\underline{n}}$. So take $B=0$. Then,

$$3A = 4$$
$$\Rightarrow \quad A = 4/3.$$

Our particular solution is then

$$y_p(x) = \frac{4}{3}xe^x .$$

Our final solution is then

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1e^x + c_2e^{-2x} + \frac{4}{3}xe^x. \end{aligned}$$

Example: Solve:

$$2y'' + 4y' + 2y = e^{-x}.$$

We already found that

$$y_h(x) = c_1e^{-x} + c_2xe^{-x} \ldots$$

So this means that guessing

$$y_p(x) = Ae^{-x} \quad ✗$$

will not work! But also, a poly$^{=}$ of degree 1 won't work either!

$$y_p(x) = (Ax + B)e^{-x} \quad ✗$$

This time we need to go up to degree 2! That is,

$$y_p(x) = (Ax^2 + Bx + C)e^{-x}.$$

But now we're super clever and know that for any B, C, Bxe^{-x} and Ce^{-x} solve the homogeneous problem, so take $B = C = 0$ and guess

$$y_p(x) = Ax^2e^{-x}.$$

$$y_p'(x) = 2Axe^{-x} - Ax^2e^{-x}$$

$$y_p''(x) = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$$

Then plug into our original eq$^{=}$:

$$\begin{aligned} e^{-x} &= 2y_p'' + 4y_p' + 2y_p \\ &= 2(2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x}) \\ &\quad + 4(2Axe^{-x} - Ax^2e^{-x}) \\ &\quad + 2(Ax^2e^{-x}) \\ &= x^2e^{-x}(\overbrace{2A - 4A + 2A}^{=0}) \\ &\quad + xe^{-x}(\underbrace{-8A + 8A}_{=0}) \\ &\quad + e^{-x}(4A) \\ &= 4Ae^{-x} \end{aligned}$$

So, it must be the case that

$$1 = 4A$$

$$\Rightarrow A = 1/4.$$

So, our particular solution is

$$y_p(x) = \frac{x^2}{4}e^{-x},$$

and our final solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1e^{-x} + c_2xe^{-x} + \frac{x^2}{4}e^{-x} \\ &= e^{-x}\left(\frac{x^2}{4} + c_2x + c_1\right). \end{aligned}$$