Lab Two

Saturday, September 14, 2019 11:53 AM

Topics:
- Linear equations
- Bernoulli equations
- Exact equations

Homework #1 due Sept. 23, 5pm.

1. Suppose we are given an ODE of the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$

for some functions a, b, c with $a(x) \neq 0$. This type of equation is a linear, first order equation, and can always be solved (in theory). First, if we divide this whole equation by $a(x)$, we can convert it to standard form:

$$\frac{dy}{dx} + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}.$$

Set
$$f(x) := \frac{b(x)}{a(x)}$$
and
$$g(x) := \frac{c(x)}{a(x)}.$$

Then, we have

$$\frac{dy}{dx} + f(x)y = g(x). \quad \circledast$$

We can then solve this equation by using an Integrating Factor and "undoing" the chain rule. Consider the integrating factor

$$\mu(x) = e^{\int f(x)}$$

Then, its derivative with respect to x:

$$\begin{aligned}\frac{d}{dx}\left(\mu(x)\right) &= e^{\int f(x)}\frac{d}{dx}\left(\int f(x)\right)\\ &= \mu(x)\frac{d}{dx}\left(\int f(x)\right)\\ &= \mu(x)f(x). \quad \circledast\circledast\end{aligned}$$

This will be useful shortly. If we multiply $\circledast$ above by $\mu(x)$, we have

$$\mu(x)\frac{dy}{dx} + \underline{\mu(x)f(x)}\,y = \mu(x)g(x).$$

From [illegible] above, we know that

$$\mu(x)f(x) = \frac{d}{dx}\left(\mu(x)\right)!$$

So,

$$\underbrace{\mu(x)\frac{dy}{dx} + \frac{d}{dx}\left(\mu(x)\right)y} = \mu(x)g(x).$$

This looks very familiar! Recall the chain rule:

$$\frac{d}{dx}\left(F(x)\,G(x)\right) = \frac{dF}{dx}G(x) + F(x)\frac{dG}{dx}.$$

This is exactly what we have above with

$$F(x) := y(x)$$
$$G(x) := \mu(x).$$

So, we can write

$$\frac{d}{dx}\left(y(x)\,\mu(x)\right) = \mu(x)g(x).$$

We can integrate both sides:

$$\int\frac{d}{dx}\left(y(x)\,\mu(x)\right) = y(x)\,\mu(x)$$
$$\|$$
$$\int\mu(x)\,g(x).$$

Hence, if we divide both sides by $e^{\mu(x)}$ we can solve for $y(x)$:

$$\boxed{y(x) = \frac{1}{\mu(x)}\int\mu(x)g(x)\,dx}$$

This is the general formula to solve any first order, linear ODE.

Solving First Order, Linear ODE's

Step 1. Convert to standard form!
Step 2. Calculate integrating factor $\mu(x)$.
Step 3. Undo the chain rule (or plug into the formula above) and solve for $y(x)$.

Examples: 1.
$$x\frac{dy}{dx} + x^2 y = x e^x. \qquad x > 0.$$

Step 1. Convert to standard form:

$$\frac{dy}{dx} + xy = e^x$$

Step 2. Find integrating factor:

$$\mu(x) = e^{\int f(x)}$$
$$= e^{\int x} = e^{x^2/2}$$

Step 3. Multiply by $\mu(x) = e^{x^2/2}$:

$$e^{x^2/2}\frac{dy}{dx} + xe^{x^2/2}y = e^{x^2/2}\cdot e^x$$

and undo the chain rule:

$$\frac{d}{dx}\left(e^{x^2/2}y\right) = e^{x^2/2+x}$$

Hence:
$$y(x) = e^{-x^2/2}\int e^{x^2/2+x}$$

Note: The integral will not always be easy!

2.
$$x\frac{dy}{dx} + 2y = x^2$$

Step 1. Standard form:

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

Step 2. Int. factor: $\mu(x) = e^{\int f(x)}$

$$= e^{\int \frac{2}{x}}$$

$$= e^{2\ln(x)}$$

Then, $e^{2\ln(x)} = e^{\ln(x^2)}$

$$= x^2.$$

Step 3. Multiply by x^2 and undo chain rule:

$$x^2\frac{dy}{dx} + 2xy = x^3$$

$$\hookrightarrow \frac{d}{dx}\left(x^2y\right) = x^3$$

$$\hookrightarrow x^2y = \int x^3$$

$$= \frac{x^4}{4} + C$$

$$\hookrightarrow y(x) = \frac{x^2}{4} + \frac{C}{x^2}, \quad x \neq 0.$$

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2. Bernoulli Equations

Suppose now that we are given a first order, nonlinear equation of the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x)y^n,$$

for some real number $n \neq 0, 1$. Then, if we divide by $a(x)$ (assume $a(x) \neq 0$), then we have

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad \circledast$$

for $P(x) := \frac{b(x)}{a(x)}$, $Q(x) := \frac{c(x)}{a(x)}$.

This is a Bernoulli Equation!

Solving Bernoulli Equations

A Bernoulli equation can be solved by using the substitution

$$u := y^{1-n}.$$

Then, if we take the derivative of u we have

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx},$$

and from the equation for y,

$$\frac{dy}{dx} = Q(x)y^n - P(x)y, \quad \text{and so}$$

plugging this into the equation for u gives:

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n}\left[Q(x)y^n - P(x)y\right] \\ &= (1-n)Q(x) - (1-n)P(x)y^{1-n} \\ &= (1-n)Q(x) - (1-n)P(x)u.\end{aligned}$$

Hence, our new equation to solve is:

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

This is a first order, linear equation! Once we solve for u, our solution to the original equation is then:

$$y(x) = [u(x)]^{\frac{1}{1-n}}, \quad n \neq 1.$$

Examples: (1) $t^2\frac{dy}{dt} + 2ty = y^3$.

Step 1. Put into standard form!

$$\frac{dy}{dx} + \frac{2}{t}y = \frac{1}{t^2}y^3$$

Step 2. Convert to a linear equation.
In this example,

$$n = 3,$$
$$P(t) = 2/t,$$
$$Q(t) = 1/t^2$$

So, solve instead

$$\frac{du}{dt} + (1-n)P(t)u = (1-n)Q(t)$$

$$\hookrightarrow \frac{du}{dt} - 2\left(\frac{2}{t}\right)u = -\frac{2}{t^2}$$

Step 3. Solve using the integrating factor!

$$\mu(t) = e^{-\int \frac{4}{t}}$$

$$= e^{-4\ln(t)}$$

So, $\mu(t) = e^{\ln(t^{-4})} = t^{-4}$.

Multply the equation by t^{-4}:

$$t^{-4}\frac{du}{dt} - 4t^{-3}u = -2t^{-6}$$

Undo the chain rule:

$$\frac{d}{dt}\left(t^{-4}u\right) = -2t^{-6}$$

Integrate:

$$\begin{aligned} t^{-4}u &= -2\int t^{-6} \\ &= -2\left[-\frac{1}{5}t^{-5}\right] + C \\ &= \frac{2}{5}t^{-5} + C \end{aligned}$$

So, the solution $u(t)$ is given by:

$$u(t) = \frac{2}{5}t^{-1} + Ct^{4}.$$

Hence, the solution $y(t)$ is given by:

$$\left.\begin{aligned} y(t) &= \left(u(t)\right)^{\frac{1}{1-n}} \\ &= \left(\frac{2}{5}t^{-1} + Ct^{4}\right)^{-1/2} \\ &= \frac{1}{\sqrt{\frac{2}{5}t^{-1} + Ct^{4}}}. \end{aligned}\right\}$$

I guess this could be "Step 4. Convert back to find $y(t)$".

(2) $\frac{dy}{dx} = \varepsilon y - \sigma y^3$.

Step 1. We're almost in standard form!

$$\frac{dy}{dx} - \varepsilon y = -\sigma y^3.$$

Step 2. Convert to a linear equation:

$n = 3$,

$P(x) = -\varepsilon$

$Q(x) = -\sigma$, so

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

$$\hookrightarrow \frac{du}{dx} + (-2)(-\varepsilon)u = (-2)(-\sigma)$$

$$\hookrightarrow \frac{du}{dx} + 2\varepsilon u = 2\sigma .$$

__Step 3.__ Solve using integrating factor:

$$\int 2\varepsilon$$

$$= 2\varepsilon x, \qquad \text{so}$$

$$\mu(x) = e^{2\varepsilon x}.$$

Multiply by $\mu(x)$ and undo chain rule:

$$e^{2\varepsilon x}\frac{du}{dx} + 2\varepsilon e^{2\varepsilon x} u = 2\sigma e^{2\varepsilon x}$$

$$\hookrightarrow \frac{d}{dx}\left(e^{2\varepsilon x} u\right) = 2\sigma e^{2\varepsilon x}$$

$$\hookrightarrow e^{2\varepsilon x} u = 2\sigma \int e^{2\varepsilon x}$$

$$= 2\sigma\left[\frac{1}{2\varepsilon} e^{2\varepsilon x}\right] + C$$

$$= \frac{\sigma}{\varepsilon} e^{2\varepsilon x} + C$$

Hence, $u(x) = \dfrac{\sigma}{\varepsilon} + Ce^{-2\varepsilon x}$.

__Step 4.__ Convert back to solve for $y(x)$:

$$y(x) = (u(x))^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\frac{\sigma}{\varepsilon} + Ce^{-2\varepsilon x}}}$$

__Exact Equations__

Suppose we are given an equation in the following form:

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

If M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the equation is called __exact__.

__Examples:__ (1) Consider the equation

$$\frac{dy}{dx} = \frac{-(y\cos x + 2xe^y)}{(\sin x + x^2e^y - 1)}.$$

Rearranging, we have:

$$\underbrace{(y\cos x + 2xe^y)}_{M(x,y)}dx + \underbrace{(\sin x + x^2e^y - 1)}_{N(x,y)}dy = 0.$$

So, $\frac{\partial M}{\partial y} = \cos x + 2xe^y$,

$\frac{\partial N}{\partial x} = \cos x + 2xe^y$.

These are equal, and so the equation is indeed exact.

(2) $$\frac{dy}{dx} = -\frac{(3xy + y^2)}{(x^2 + xy)}$$

Rearranging, we have

$$\underbrace{(3xy + y^2)}_{M(x,y)}dx + \underbrace{(x^2 + xy)}_{N(x,y)}dy = 0$$

So,

$\frac{\partial M}{\partial y} = 3x + 2y$,

$\frac{\partial N}{\partial x} = 2x + y$.

This time they are NOT equal, and so the equation is not exact!

Solving Exact Equations

When an equation is exact, the method to solve the equation relies on the fact that

$$\psi_{xy} = \psi_{yx}$$

$$\left(\text{or } \frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x}\right)$$

for "nice" functions $\psi(x,y)$.

So, we seek a function $\psi(x,y)$ so that

$$\begin{cases} \psi_x = M(x,y) \\ \psi_y = N(x,y), \end{cases}$$

since $\frac{\partial N}{\partial x} = \psi_{yx}$

$= \psi_{xy} = \frac{\partial M}{\partial y}$.

So, if $\psi_x = M(x,y)$, we integrate in x:

$$\psi(x,y) = \int \frac{\partial \psi}{\partial x}\, dx$$

$$= \int M(x,y)\, dx \;.$$

Since M depends on both x and y, when we integrate, our constant term is now a function of y instead! We can then take the derivative of $\psi(x,y)$ with respect to y in order to solve for this mystery function. Let's do an example to clarify. If we have

$$(2x+3)dx + (2y-2)dy = 0,$$

we see that

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x},$$

and so the equation is exact. So, let's integrate $M(x,y) = 2x+3$ in x:

$$\int M(x,y)dx = \int (2x+3)dx$$

$$= x^2 + 3x + h(y),$$

where we need to find $h(y)$! So, we have that

$$\psi(x,y) = x^2 + 3x + h(y),$$

and we know that $\psi(x,y)$ must satisfy

$$\begin{cases} \psi_x = M(x,y) \\ \psi_y = N(x,y) \end{cases}$$

Since we have already used the first equation, we will now use the second equation. So,

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3x + h(y) \right)$$

$$= h'(y).$$

But, we also know that

$$\frac{\partial \psi}{\partial y} = N(x,y).$$

So, it must then be true that

$$h'(y) = \frac{\partial \psi}{\partial y} = N(x,y), \text{ or}$$

$$h'(y) = 2y - 2.$$

Integrating with respect to y gives us

$$\begin{aligned} h(y) &= \int h'(y)\,dy \\ &= \int 2y-2 \\ &= y^2 - 2y. \end{aligned}$$

Thus, our final solution is given (implicitely) by

$$\begin{aligned} \psi(x,y) &= x^2 + 3x + h(y) \\ &= x^2 + 3x + y^2 - 2y. \end{aligned}$$

Examples: (1) Let's go back to a previous example:

$$\underbrace{(y\cos x + 2xe^y)}_{M(x,y)}dx + \underbrace{(\sin x + x^2e^y - 1)}_{N(x,y)}dy = 0.$$

Step 1. Check if it is an exact equation (we have already done this).

Step 2. Look for $\psi(x,y)$ that satisfies

$$\begin{cases} \psi_x = M(x,y) \\ \psi_y = N(x,y) \end{cases}$$

First, integrate the first equation in x:

$$\begin{aligned} \psi(x,y) &= \int \psi_x\,dx \\ &= \int M(x,y)\,dx \\ &= \int (y\cos x + 2xe^y)\,dx \\ &= y\sin x + x^2e^y + h(y). \end{aligned}$$

Then, use the second equation to find

$h(y)$: $\quad \dfrac{\partial \psi}{\partial y} = N(x,y)$, so

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= \frac{\partial}{\partial y}\left(y\sin x + x^2 e^y + h(y) \right) \\
&= \sin x + x^2 e^y + h'(y) \\
&= N(x,y) \\
&= \sin x + x^2 e^y - 1
\end{aligned}$$

So,

$$\cancel{\sin x} \cancel{+ x^2 e^y} + h'(y) = \cancel{\sin x} + \cancel{x^2 e^y} - 1$$

$$\hookrightarrow \quad h'(y) = -1.$$

Integrate in y to obtain:

$$\begin{aligned}
h(y) &= \int h'(y)\,dy \\
&= -\int dy = -y.
\end{aligned}$$

Step 3. Put together the final solution:

$$\begin{aligned}
\psi(x,y) &= y\sin x + x^2 e^y + h(y) \\
&= y\sin x + x^2 e^y - y \\
&= y(\sin x - 1) + x^2 e^y.
\end{aligned}$$